Mass in

Kähler Geometry

Claude LeBrun
Stony Brook University

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Joint work with
Joint work with

Hans-Joachim Hein
University of Maryland
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Definition. A complete, non-compact Riemannian $n$-manifold $(M^n, g)$
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g_{jk} = \delta_{jk} + O(|x|^{1 - \frac{n}{2} - \varepsilon})
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Definition. The mass (at a given end) of an ALE $n$-manifold is defined to be

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Seems to depend on choice of coordinates!
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**Bartnik (1986):** With weak fall-off conditions, the mass is well-defined & coordinate independent.
Motivation:
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When \( n = 3 \), ADM mass in general relativity.
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Reads off “apparent mass” from strength of the gravitational field far from an isolated source.
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also has mass $m$. 
Positive Mass Conjecture:
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Physical intuition:
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Local matter density $\geq 0$
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Local matter density $\geq 0 \implies$ total mass $\geq 0$. 
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Conjectured true in ALE case, too.
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Scalar-flat Kähler metrics
on line bundles \( L \to \mathbb{CP}_1 \) of Chern-class \( \leq -3 \).
Mass of ALE Kähler manifolds?
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Scalar-flat Kähler case?
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Lemma.
Mass of ALE Kähler manifolds?

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Lemma. Any ALE Kähler manifold
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Lemma. Any ALE Kähler manifold has only one end.
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\[ n = 2m \geq 4 \]
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Several different proofs are known.
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**Lemma.** *Any ALE Kähler manifold has only one end.*

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each end is pseudo-convex at infinity.
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Another is more topological:
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**Lemma.** Any ALE Kähler manifold has only one end.

Several different proofs are known.

Several are analytic:

each end is pseudo-convex at infinity.

Another is more topological:

intersection form on $H^2$ of compactification.
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**Lemma.** Any ALE Kähler manifold has only one end.
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\textbf{Lemma.} \textit{Any ALE Kähler manifold has only one end.}

\textbf{Upshot:}
Mass of \textbf{ALE Kähler} manifolds?

Scalar-flat Kähler case?

\textbf{Lemma.} Any \textit{ALE Kähler manifold has only one end.}

\hline

\textbf{Upshot:}

Mass of an \textit{ALE Kähler manifold} is unambiguous.
Mass of **ALE Kähler** manifolds?

**Scalar-flat Kähler case?**

**Lemma.** *Any ALE Kähler manifold has only one end.*

**Upshot:**

Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!
We begin with the scalar-flat Kähler case.
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**Theorem A.**
We begin with the scalar-flat Kähler case.

Theorem A. The mass
We begin with the scalar-flat Kähler case.

**Theorem A.** The mass of an ALE
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**Theorem A.** The mass of an ALE scalar-flat Kähler manifold.
We begin with the scalar-flat Kähler case.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!
The explicit formula reproduces the mass in cases where it previously had been laboriously computed from the definition.
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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)
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---

Here

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H^p_c(M) := \frac{\text{ker } d : \mathcal{E}^p_c(M) \to \mathcal{E}^{p+1}_c(M)}{d\mathcal{E}^{p-1}_c(M)}
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induced by the inclusion of compactly supported smooth forms into all forms.
We can now state our mass formula:

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**Theorem C.** *Any ALE Kähler manifold $(M, g, J)$ of complex dimension $m$ has mass given by*

\[
m(M, g) = -\left\langle \frac{c_1}{2m-1} \right\rangle \left( 2m - 1 \right) \pi^{m-1} + \frac{(m-1)!}{4(2m-1)} \left( \int_M s \, g \, d\mu_g \right)
\]
We can now state our mass formula:

**Theorem C.** Any ALE Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\left\langle c_1, \omega \right\rangle_{m-1} (2^m - 1) \pi^{m-1} + \frac{(m-1)!}{4 (2^m - 1) \pi^m} \int_M s_g d\mu_g
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- \(\langle , \rangle\) is pairing between \(H^2_c(M)\) and \(H^{2m-2}(M)\).
\[ m(M, g) = -\frac{\langle \clubsuit (c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g \]
\[
\frac{(2m - 1)\pi^m}{(m - 1)!} m(M, g) = -\frac{4\pi}{(m - 1)!} \langle \clubsuit(c_1), [\omega]^{m-1} \rangle + \int_M s g d\mu_g
\]
For a compact Kähler manifold \((M^{2m}, g, J)\),

\[
\int_M s_g d\mu_g = \frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle
\]
For a compact Kähler manifold \((M^{2m}, g, J)\),

\[
0 = -\frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle + \int_M s_g d\mu_g
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For an ALE Kähler manifold $(M^{2m}, g, J)$,

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\]

So the mass is a “boundary correction” to the topological formula for the total scalar curvature.
Theorem C. Any ALE Kähler manifold $(M, g, J)$ of complex dimension $m$ has mass given by

$$m(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g$$
Corollary. Any ALE scalar-flat Kähler manifold $(M, g, J)$ of complex dimension $m$ has mass given by

$$m(M, g) = -\frac{\langle \clubsuit (c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}}.$$
Corollary. Any ALE scalar-flat Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle \clubsuit (c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}}.
\]

So Theorem A is an immediate consequence!
Theorem D.
Theorem D (Positive Mass Theorem). Any AE Kähler manifold with
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Theorem D (Positive Mass Theorem). Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:

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Moreover, $m = 0 \iff$
**Theorem D** (Positive Mass Theorem). Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:

\[ \text{AE \& Kähler \& } s \geq 0 \implies m(M, g) \geq 0. \]

Moreover, \( m = 0 \iff (M, g) \) is Euclidean space.
**Theorem D** (Positive Mass Theorem). Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:

\[ AE & \text{Kähler} & s \geq 0 \quad \implies \quad m(M, g) \geq 0. \]

Moreover, \( m = 0 \iff (M, g) \) is Euclidean space.

Proof actually shows something stronger!
Theorem E.

Let \((M, g, J)\) be an AE Kähler manifold with scalar curvature \(s \geq 0\). Then \((M, J)\) carries a canonical divisor \(D\) that is expressed as a sum \(\sum n_j D_j\) of compact complex hypersurfaces with positive integer coefficients, with the property that \(\bigcup n_j D_j \neq \emptyset\) whenever \((M, J) \neq \mathbb{C}^m\). In terms of this divisor, we then have
Theorem E (Penrose Inequality).

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with \(= \iff (M, g, J)\) is scalar-flat Kähler.