Zoll Manifolds,

Complex Surfaces, &

Holomorphic Disks, II

Claude LeBrun
Stony Brook University

Autumn School on Holomorphic Disks
Schloss Rauischholzhausen, November 15, 2018

Joint work with

Lionel Mason
Oxford University

Definition. *A Zoll metric on a smooth manifold M is a Riemannian metric g whose geodesics are all simple closed curves of equal length.*

Definition. *A Zoll metric on a smooth manifold M is a Riemannian metric g whose geodesics are all simple closed curves of equal length.*

Simple closed curve: embedded circle.

Definition. *A Zoll metric on a smooth manifold M is a Riemannian metric g whose geodesics are all simple closed curves of equal length.*

Definition. *A Zoll metric on a smooth manifold M is a Riemannian metric g whose geodesics are all simple closed curves of equal length.*

Definition. *Let M be a smooth compact manifold. A Zoll conection on M is a torsion-free affine connection ∇ for which the image of each maximally-extended geodesic is a simple closed curve.*

Definition. *A Zoll metric on a smooth manifold M is a Riemannian metric g whose geodesics are all simple closed curves of equal length.*

Definition. *Let M be a smooth compact manifold. A Zoll conection on M is a torsion-free affine connection ∇ for which the image of each maximally-extended geodesic is a simple closed curve.*

Definition. *A Zoll projective structure $[\nabla]$ on M is the projective equivalence class of some Zoll connection ∇.*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface,*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is* *tame* *if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

DBA Epstein: Any foliation of a compact 3-manifold by circles is a Seifert fibration.

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

DBA Epstein: Any foliation of a compact 3-manifold by circles is a Seifert fibration.

Oriented case: any geodesic has nbhd $\approx \mathbb{R}^2 - \{0\}$.

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

DBA Epstein: Any foliation of a compact 3-manifold by circles is a Seifert fibration.

Oriented case: any geodesic has nbhd $\approx \mathbb{R}^2 - \{0\}$.

Now use Jordan curve theorem.

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

DBA Epstein: Any foliation of a compact 3-manifold by circles is a Seifert fibration.

Oriented case: any geodesic has nbhd $\approx \mathbb{R}^2 - \{0\}$.

Now use Jordan curve theorem.

Non-oriented case via double cover.

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Proposition. *If* $(M, [\nabla])$ *is Zoll and tame,*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Proposition. *If* $(M, [\nabla])$ *is Zoll and tame, then* M *is compact.*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Proposition. *If* $(M, [\nabla])$ *is Zoll and tame, then* M *is compact.*

Proposition. *If* $(M, [\nabla])$ *is Zoll and tame,*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Proposition. *If* $(M, [\nabla])$ *is Zoll and tame, then* M *is compact.*

Proposition. *If* $(M, [\nabla])$ *is Zoll and tame, then so is its universal cover* $(\tilde{M}, [\tilde{\nabla}])$*.*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Theorem. *If a compact surface* M^2

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Theorem. *If a compact surface* M^2 *admits a Zoll projective connection* $[\nabla]$*,*

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Theorem. *If a compact surface* M^2 *admits a Zoll projective connection* $[\nabla]$*, then*

$$|\pi_1(M)| < \infty,$$

Definition. *A Zoll projective connection* $[\nabla]$ *on a smooth manifold* M *is tame if the associated foliation of* $\mathbb{P}(TM)$ *by circles is locally trivial.*

Theorem. *If* M^2 *is a compact surface, then any Zoll projective connection* $[\nabla]$ *on* M *is tame.*

Theorem. *If a compact surface* M^2 *admits a Zoll projective connection* $[\nabla]$*, then*

$$|\pi_1(M)| < \infty,$$

and hence

$$M \approx S^2 \text{ or } \mathbb{RP}^2.$$

Proposition.

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*,*

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

$$\mathbb{P}TM$$

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

$$\begin{array}{ccc} & \mathbb{P}TM & \\ \mu \swarrow & & \\ M & & \end{array}$$

Proposition. *For any Zoll $[\nabla]$ on a compact surface M^2, we have a double fibration*

$$\begin{array}{ccc} & \mathbb{P}TM & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ M & & N \end{array}$$

Proposition. *For any Zoll $[\nabla]$ on a compact surface M^2, we have a double fibration*

$$\begin{array}{ccc} & \mathbb{P}TM & \\ \mu \swarrow & & \searrow \nu \\ M & & N \end{array}$$

where $N \approx \mathbb{RP}^2$ is the space of geodesics of $[\nabla]$.

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

$$\begin{array}{ccc} & \mathbb{P}TM & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ M & & N \end{array}$$

where $N \approx \mathbb{RP}^2$ *is the space of geodesics of* $[\nabla]$.

If $M \approx \mathbb{RP}^2$, $\nu : \mathbb{P}TM \to N$ *can be identified with* $\mathbb{P}TN \to N$ *via* $\nu_*(\ker \mu_*)$.

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

$$\begin{array}{ccc} & \mathbb{P}TM & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ M & & N \end{array}$$

where $N \approx \mathbb{RP}^2$ *is the space of geodesics of* $[\nabla]$.

If $M \approx \mathbb{RP}^2$, $\nu : \mathbb{P}TM \to N$ *can be identified with* $\mathbb{P}TN \to N$ *via* $\nu_*(\ker \mu_*)$.

If $M \approx S^2$, $\nu : \mathbb{P}TM \to N$ *can be identified with* $\mathbb{S}TN \to N$.

And now for something completely different...

Proposition.

Proposition. *Let* M^2 *be any surface,*

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

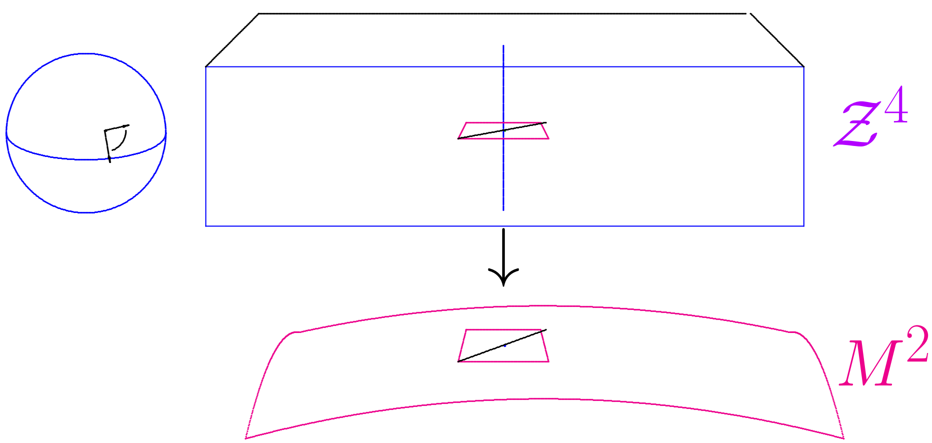
Then any affine connection ∇ *on* M

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$

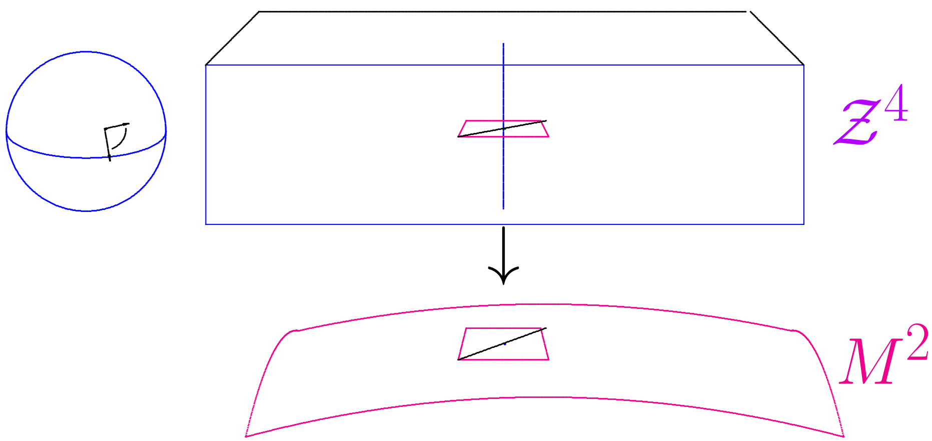
Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$



Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$



$$\mathbf{D} = \mathbf{L}_1 \oplus \mathbf{L}_2 \subset T_{\mathbb{C}}\mathcal{Z}$$

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$

$$\left[\left(\frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} \right) \Big|_{(x^1, x^2)} \right] \longleftrightarrow (x^1, x^2, \zeta).$$

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$

$$\left[\left(\frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2}\right)\Big|_{(x^1,x^2)}\right] \longleftrightarrow (x^1, x^2, \zeta).$$

$$\mathbf{D} = \operatorname{span}\left\{ \Xi\,, \ \frac{\partial}{\partial \overline{\zeta}} \right\}$$

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$

$$\left[\left(\frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} \right) \Big|_{(x^1, x^2)} \right] \longleftrightarrow (x^1, x^2, \zeta).$$

$$\mathbf{D} = \operatorname{span} \left\{ \Xi \, , \quad \frac{\partial}{\partial \bar{\zeta}} \right\}$$

$$\Xi = \frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} + P(x, \zeta) \frac{\partial}{\partial \zeta},$$

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$

$$\left[\left(\frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} \right) \Big|_{(x^1, x^2)} \right] \longleftrightarrow (x^1, x^2, \zeta).$$

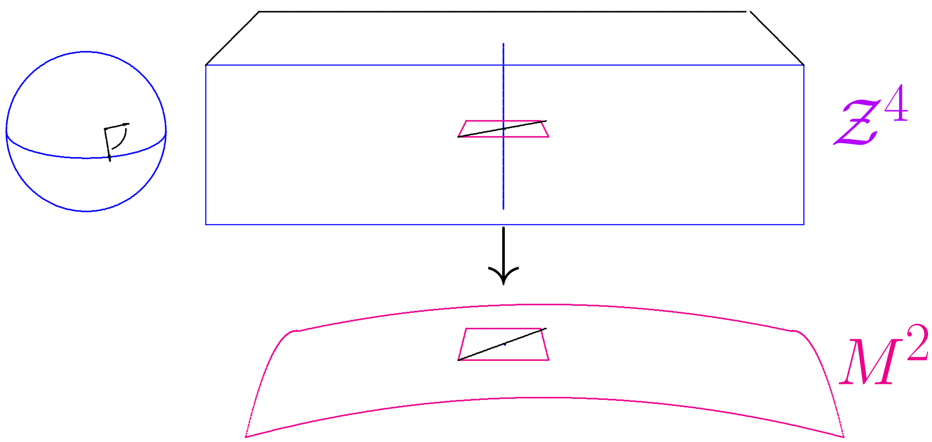
$$\mathbf{D} = \text{span} \left\{ \Xi , \quad \frac{\partial}{\partial \bar{\zeta}} \right\}$$

$$\Xi = \frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} + P(x, \zeta) \frac{\partial}{\partial \zeta},$$

$$P = -\Gamma^2_{11} + \left[\Gamma^1_{11} - 2\Gamma^2_{12} \right] \zeta + \left[2\Gamma^1_{12} - \Gamma^2_{22} \right] \zeta^2 + \Gamma^1_{22} \zeta^3$$

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$



$$\mathbf{D} = \mathbf{L}_1 \oplus \mathbf{L}_2 \subset T_{\mathbb{C}}\mathcal{Z}$$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

$$\mathbf{D} = \text{span}\left\{ \Xi\, , \quad \frac{\partial}{\partial \overline{\zeta}} \right\}$$

$$\Xi = \frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} + P(x, \zeta) \frac{\partial}{\partial \zeta},$$

$$P = -\Gamma_{11}^2 + \left[\Gamma_{11}^1 - 2\Gamma_{12}^2\right]\zeta + \left[2\Gamma_{12}^1 - \Gamma_{22}^2\right]\zeta^2 + \Gamma_{22}^1\zeta^3$$

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ with

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

Newlander-Nirenberg Theorem

Newlander-Nirenberg Theorem

(Nijenhuis-Wood/Malgrange version)

Newlander-Nirenberg Theorem

(Nijenhuis-Wood/Malgrange version)

Complex structure J on $\mathfrak{X}^{2m}$

Newlander-Nirenberg Theorem

(Nijenhuis-Wood/Malgrange version)

Complex structure J on $\mathfrak{X}^{2m}$

$$\Longleftrightarrow$$

$C^{1,\alpha}$ rank-m sub-bundle $T^{0,1} \subset T_{\mathbb{C}}\mathfrak{X}$

Newlander-Nirenberg Theorem

(Nijenhuis-Wood/Malgrange version)

Complex structure J on $\mathfrak{X}^{2m}$

$$\Longleftrightarrow$$

$C^{1,\alpha}$ rank-m sub-bundle $T^{0,1} \subset T_{\mathbb{C}}\mathfrak{X}$ with

$$T^{0,1} \cap \overline{T^{0,1}} = 0$$

Newlander-Nirenberg Theorem

(Nijenhuis-Wood/Malgrange version)

Complex structure J on $\mathfrak{X}^{2m}$

$$\Longleftrightarrow$$

$C^{1,\alpha}$ rank-m sub-bundle $T^{0,1} \subset T_{\mathbb{C}}\mathfrak{X}$ with

$$T^{0,1} \cap \overline{T^{0,1}} = 0$$

and

$$[C^1(T^{0,1}), C^1(T^{0,1})] \subset C^0(T^{0,1}).$$

Newlander-Nirenberg Theorem

(Hill-Taylor improvement)

Complex structure J on $\mathfrak{X}^{2m}$

$$\Longleftrightarrow$$

C^1 rank-m sub-bundle $T^{0,1} \subset T_{\mathbb{C}}\mathfrak{X}$ with

$$T^{0,1} \cap \overline{T^{0,1}} = 0$$

and

$$[C^1(T^{0,1}), C^1(T^{0,1})] \subset C^0(T^{0,1}).$$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

and

$$\dim \mathbf{D}_z \cap \overline{\mathbf{D}}_z = \begin{cases} 0 & \text{if } z \notin \mathbb{P}TM, \\ 1 & \text{if } z \in \mathbb{P}TM. \end{cases}$$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

and

$$\dim \mathbf{D}_z \cap \overline{\mathbf{D}}_z = \begin{cases} 0 & \textit{if } z \not\in \mathbb{P}TM, \\ 1 & \textit{if } z \in \mathbb{P}TM. \end{cases}$$

Moreover, two connections ∇ *and* $\hat{\nabla}$ *give rise to the same* $\mathbf{D}$ *iff they are projectively equivalent.*

Newlander-Nirenberg Theorem

(Hill-Taylor improvement)

Complex structure J on $\mathfrak{X}^{2m}$

$$\Longleftrightarrow$$

C^1 rank-m sub-bundle $T^{0,1} \subset T_{\mathbb{C}}\mathfrak{X}$ with

$$T^{0,1} \cap \overline{T^{0,1}} = 0$$

and

$$[C^1(T^{0,1}), C^1(T^{0,1})] \subset C^0(T^{0,1}).$$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

and

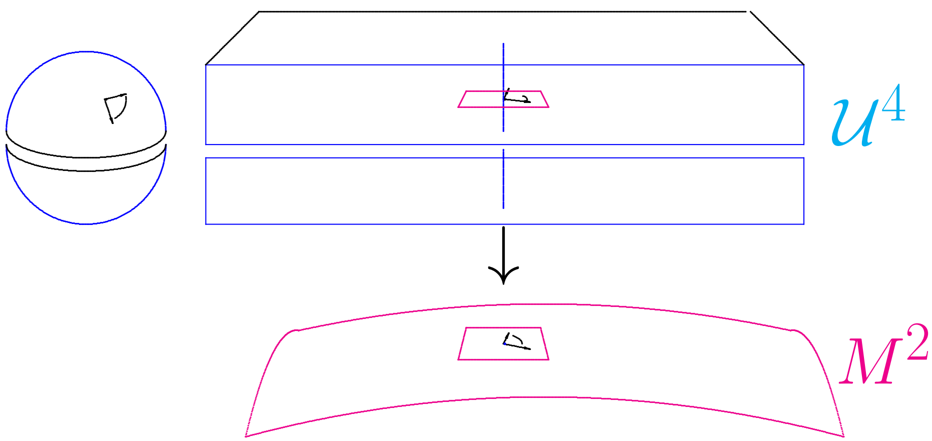
$$\dim \mathbf{D}_z \cap \overline{\mathbf{D}}_z = \begin{cases} 0 & \textit{if } z \notin \mathbb{P}TM, \\ 1 & \textit{if } z \in \mathbb{P}TM. \end{cases}$$

Moreover, two connections ∇ *and* $\hat{\nabla}$ *give rise to the same* $\mathbf{D}$ *iff they are projectively equivalent.*

Corollary. *For any* $(M^2, [\nabla])$,

$$\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$$

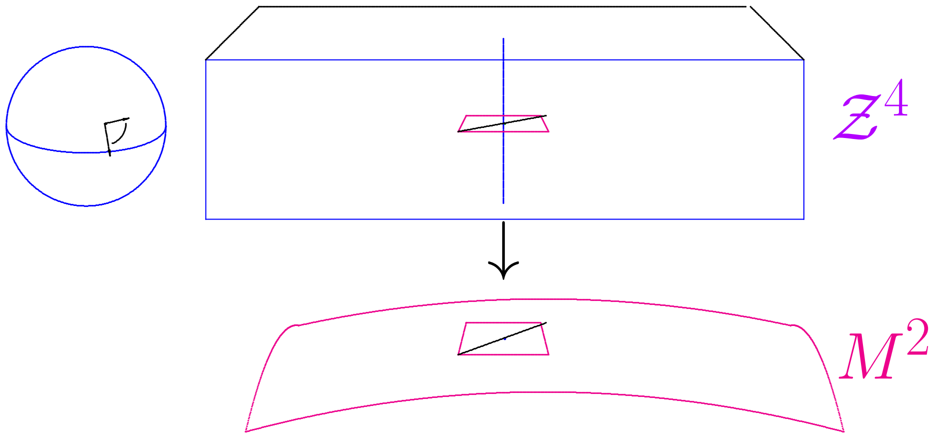
is a complex manifold.



Corollary. *For any* $(M^2, [\nabla])$,

$$\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$$

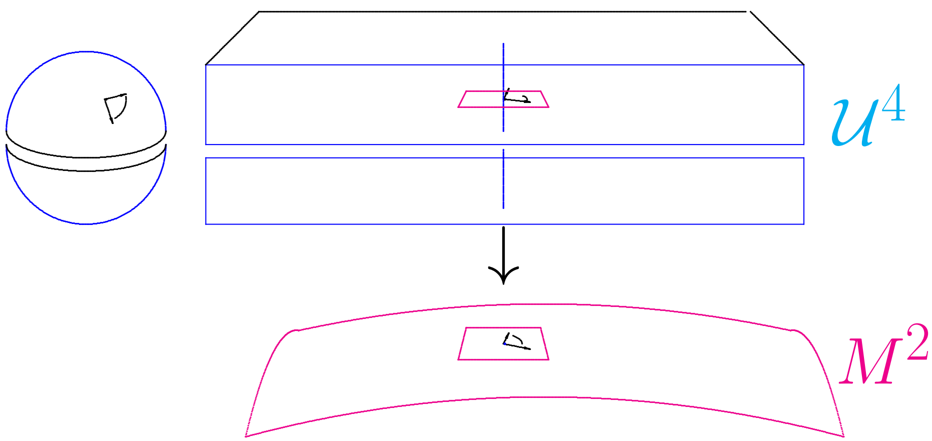
is a complex manifold.



Corollary. *For any* $(M^2, [\nabla])$,

$$\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$$

is a complex manifold.



Newlander-Nirenberg Theorem

(Hill-Taylor improvement)

Complex structure J on $\mathfrak{X}^{2m}$

$$\Longleftrightarrow$$

C^1 rank-m sub-bundle $T^{0,1} \subset T_{\mathbb{C}}\mathfrak{X}$ with

$$T^{0,1} \cap \overline{T^{0,1}} = 0$$

and

$$[C^1(T^{0,1}), C^1(T^{0,1})] \subset C^0(T^{0,1}).$$

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\xi u = 0.$$

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$[\xi, \eta]u = 0.$$

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\xi u = 0.$$

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\frac{\partial}{\partial \bar{z}^j} u = 0.$$

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\xi u = 0.$$

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\frac{\partial}{\partial \bar{z}^j} u = 0.$$

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\frac{\partial}{\partial \bar{z}^j} u = 0.$$

Real analog: Frobenius theorem

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\frac{\partial}{\partial \bar{z}^j} u = 0.$$

Real analog: Frobenius theorem

Functions constant along leaves of a foliation.

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\frac{\partial}{\partial \bar{z}^j} u = 0.$$

Real analog: Frobenius theorem

Functions constant along leaves of a foliation.

Real case involves global questions, too.

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\frac{\partial}{\partial \bar{z}^j} u = 0.$$

Real analog: Frobenius theorem

Functions constant along leaves of a foliation.

Real case involves global questions, too.

Is leaf space a manifold?

Why integrability condition?

Holomorphic functions killed by elements of $T^{0,1}$.

$$\frac{\partial}{\partial \bar{z}^j} u = 0.$$

Real analog: Frobenius theorem

Functions constant along leaves of a foliation.

Real case involves global questions, too.

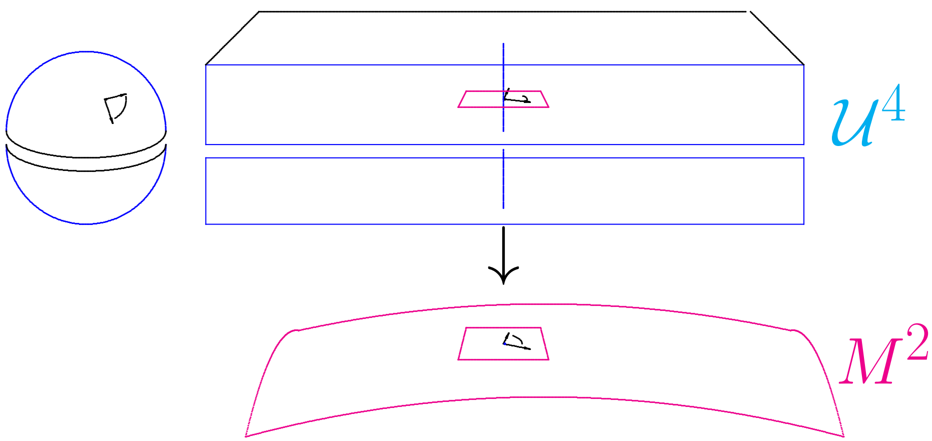
Is leaf space a manifold?

Our situation: mixture of real and complex cases!

Corollary. *For any* $(M^2, [\nabla])$,

$$\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$$

is a complex manifold.



Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

and

$$\dim \mathbf{D}_z \cap \overline{\mathbf{D}}_z = \begin{cases} 0 & \text{if } z \notin \mathbb{P}TM, \\ 1 & \text{if } z \in \mathbb{P}TM. \end{cases}$$

Proposition. *Let M^2 be any surface, and let $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ be its projectivized complexified tangent bundle.*

Then any affine connection ∇ on M determines a rank-2 sub-bundle $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ with

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

and

$$\dim \mathbf{D}_z \cap \overline{\mathbf{D}}_z = \begin{cases} 0 & \textit{if } z \not\in \mathbb{P}TM, \\ 1 & \textit{if } z \in \mathbb{P}TM. \end{cases}$$

Functions killed by $\mathbf{D}$ across real slice?

Need to blow down $\mathbb{P}TM$!

Need to blow down $\mathbb{P}TM$!

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

$$\begin{array}{ccc} & \mathbb{P}TM & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ M & & N \end{array}$$

where $N \approx \mathbb{RP}^2$ *is the space of geodesics of* $[\nabla]$.

Need to blow down $\mathbb{P}TM$!

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2, *we have a double fibration*

$$\begin{array}{ccccc} & & \mathbb{P}TM & & \\ & \overset{\mu}{\swarrow} & & \overset{\nu}{\searrow} & \\ M & & & & N \end{array}$$

where $N \approx \mathbb{RP}^2$ *is the space of geodesics of* $[\nabla]$.

If $M \approx \mathbb{RP}^2$, $\nu : \mathbb{P}TM \to N$ *can be identified with* $\mathbb{P}TN \to N$ *via* $\nu_*(\ker \mu_*)$.

Need to blow down $\mathbb{P}TM$!

- If $M \approx \mathbb{RP}^2$, let $\mathcal{N}^4$ be obtained from $\mathcal{Z}^4$

Need to blow down $\mathbb{P}TM$!

- If $M \approx \mathbb{RP}^2$, let $\mathcal{N}^4$ be obtained from $\mathcal{Z}^4$ by collapsing $\mathbb{P}TM$ to $\mathcal{N}$ along the fibers of ν.

Need to blow down $\mathbb{P}TM$!

- If $M \approx \mathbb{RP}^2$, let $\mathcal{N}^4$ be obtained from $\mathcal{Z}^4$ by collapsing $\mathbb{P}TM$ to $\mathcal{N}$ along the fibers of ν. Let

$$\Psi : \mathcal{Z} \to \mathcal{N}$$

be the tautological blowing down map.

Need to blow down $\mathbb{P}TM$!

Proposition. *For any Zoll* $[\nabla]$ *on a compact surface* M^2*, we have a double fibration*

$$\begin{array}{ccccc} & & \mathbb{P}TM & & \\ & \mu \swarrow & & \searrow \nu & \\ M & & & & N \end{array}$$

where $N \approx \mathbb{RP}^2$ *is the space of geodesics of* $[\nabla]$.

If $M \approx \mathbb{RP}^2$, $\nu : \mathbb{P}TM \to N$ *can be identified with* $\mathbb{P}TN \to N$ *via* $\nu_*(\ker \mu_*)$.

Need to blow down $\mathbb{P}TM$!

Proposition. *For any Zoll $[\nabla]$ on a compact surface M^2, we have a double fibration*

$$\begin{array}{ccc} & \mathbb{P}TM & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ M & & N \end{array}$$

where $N \approx \mathbb{RP}^2$ is the space of geodesics of $[\nabla]$.

If $M \approx \mathbb{RP}^2$, $\nu : \mathbb{P}TM \to N$ can be identified with $\mathbb{P}TN \to N$ via $\nu_*(\ker \mu_*)$.

If $M \approx S^2$, $\nu : \mathbb{P}TM \to N$ can be identified with $\mathbb{S}TN \to N$.

Need to blow down $\mathbb{P}TM$!

- If $M \approx S^2$,

Need to blow down $\mathbb{P}TM$!

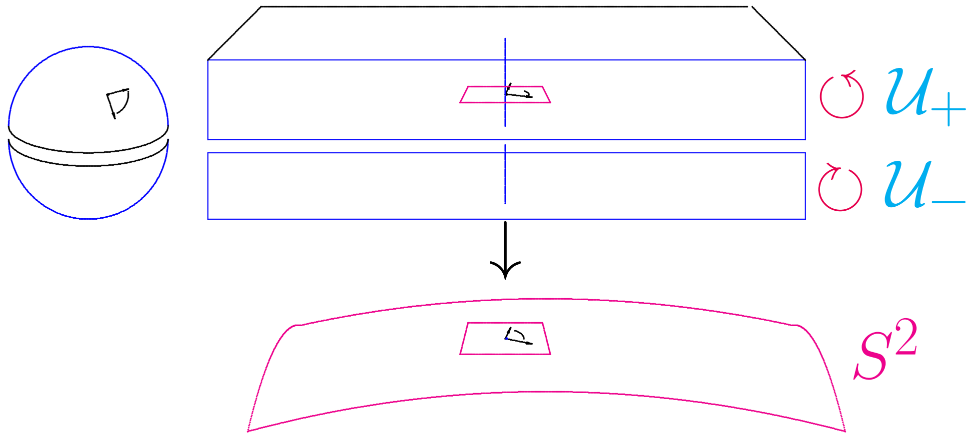
- If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$

Need to blow down $\mathbb{P}TM$!

- If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_\pm$:

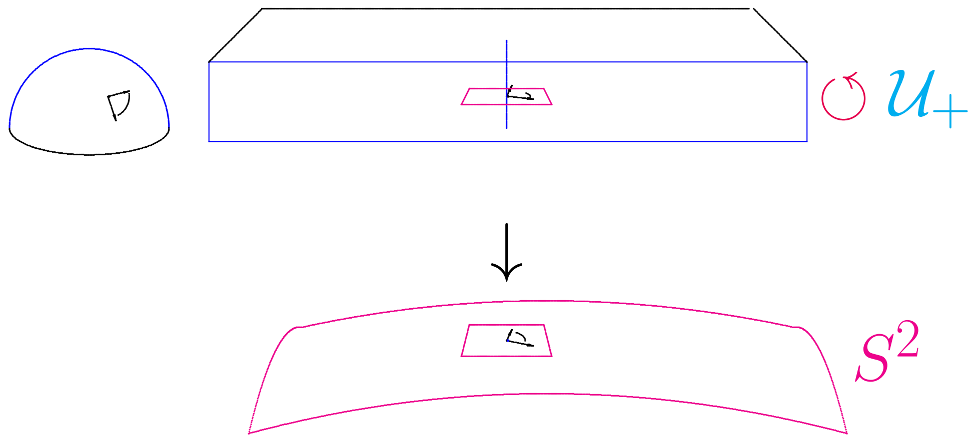
Need to blow down $\mathbb{P}TM$!

• If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_{\pm}$:



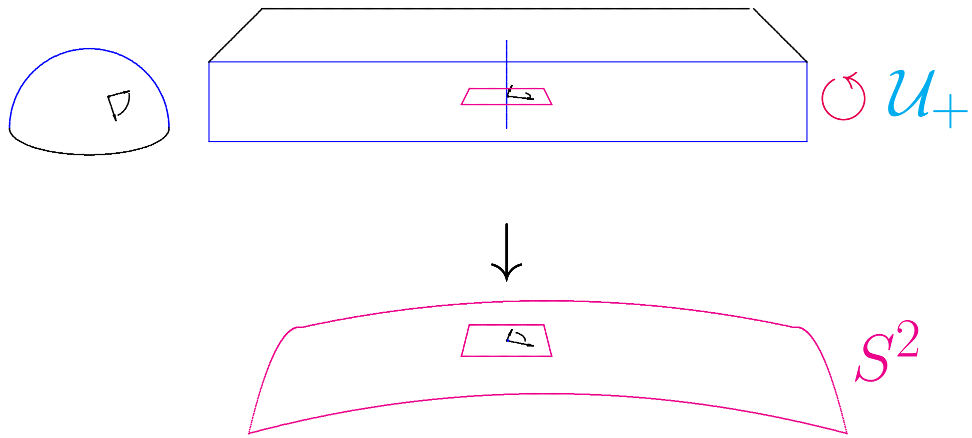
Need to blow down $\mathbb{P}TM$!

- If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_\pm$:



Need to blow down $\mathbb{P}TM$!

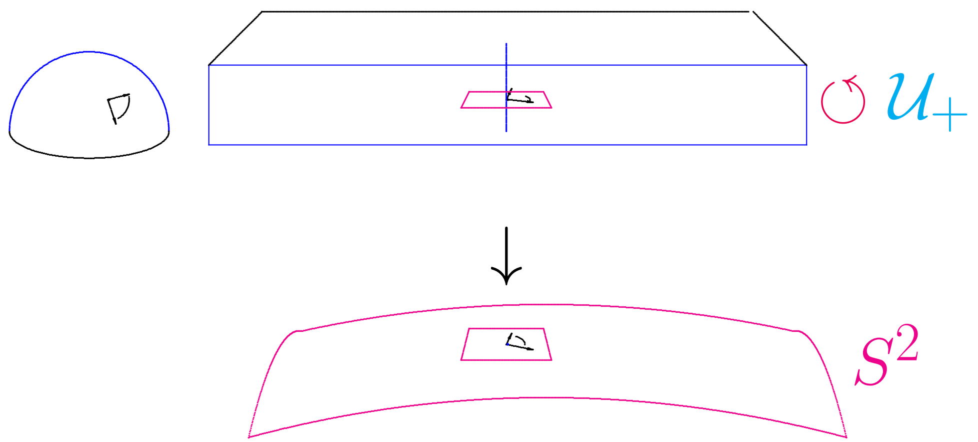
- If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_\pm$:



Let $\mathcal{Z}_+ = \mathcal{U}_+ \cup \mathbb{P}TM$,

Need to blow down $\mathbb{P}TM$!

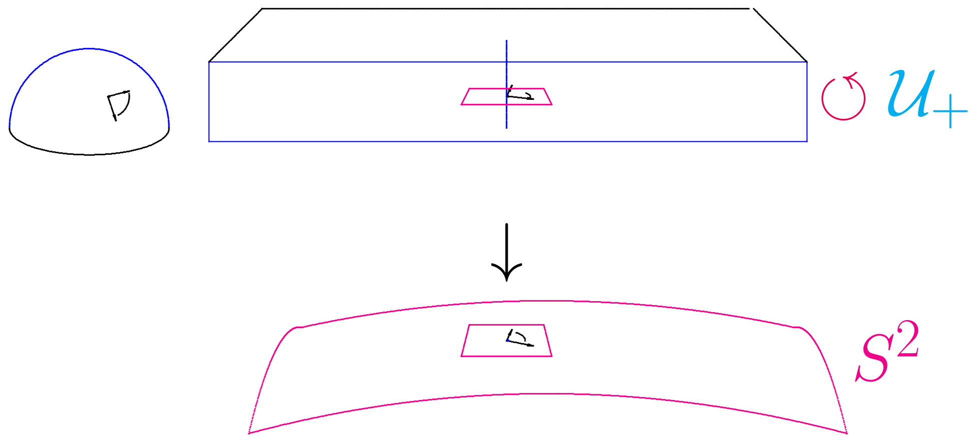
• If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_\pm$:



Let $\mathcal{Z}_+ = \mathcal{U}_+ \cup \mathbb{P}TM$, and let $\mathcal{N}$ be obtained from $\mathcal{Z}_+$

Need to blow down $\mathbb{P}TM$!

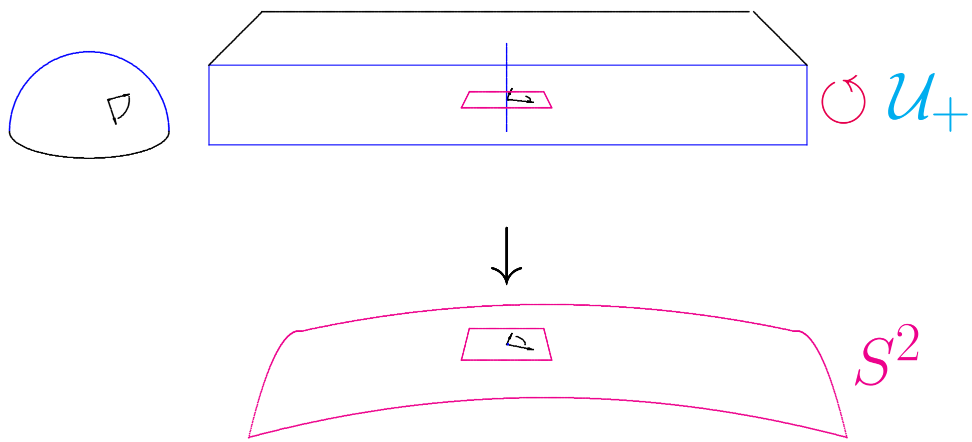
- If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_\pm$:



Let $\mathcal{Z}_+ = \mathcal{U}_+ \cup \mathbb{P}TM$, and let $\mathcal{N}$ be obtained from $\mathcal{Z}_+$ by collapsing $\partial\mathcal{Z}_+ = \mathbb{P}TM$ to N.

Need to blow down $\mathbb{P}TM$!

- If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_\pm$:



Let $\mathcal{Z}_+ = \mathcal{U}_+ \cup \mathbb{P}TM$, and let $\mathcal{N}$ be obtained from $\mathcal{Z}_+$ by collapsing $\partial\mathcal{Z}_+ = \mathbb{P}TM$ to N. Let

$$\Psi : \mathcal{Z}_+ \to \mathcal{N}$$

be the blowing down map.

Proposition.

Proposition. *For any C^3 compact Zoll surface* $(M^2, [\nabla])$,

Proposition. *For any* C^3 *compact Zoll surface* $(M^2, [\nabla])$, *there is a unique almost-complex structure* J

Proposition. *For any* C^3 *compact Zoll surface* $(M^2, [\nabla])$, *there is a unique almost-complex structure* J *on* $\mathcal{N}^4$

Proposition. *For any* C^3 *compact Zoll surface* $(M^2, [\nabla])$, *there is a unique almost-complex structure* J *on* $\mathcal{N}^4$ *such that*

$$\Psi_*(\mathbf{D}) \subset T^{0,1}(\mathcal{N}, J).$$

Proposition. *For any* C^3 *compact Zoll surface* $(M^2, [\nabla])$, *there is a unique almost-complex structure* J *on* $\mathcal{N}^4$ *such that*

$$\Psi_*(\mathbf{D}) \subset T^{0,1}(\mathcal{N}, J).$$

Moreover, J *is integrable, and so makes* $\mathcal{N}$ *into a compact complex surface.*

Proposition. *For any C^3 compact Zoll surface $(M^2, [\nabla])$, there is a unique almost-complex structure J on $\mathcal{N}^4$ such that*

$$\Psi_*(\mathbf{D}) \subset T^{0,1}(\mathcal{N}, J).$$

Moreover, J is integrable, and so makes $\mathcal{N}$ into a compact complex surface.

$[\nabla]$	J	Integrability Theorem
C^{14}	C^4	Newlander-Nirenberg (1957)
C^{10}	C^2	Malgrange (1968)
C^3	Lipschitz	Hill-Taylor (2002)

End, Part II