Zoll Manifolds,

Complex Surfaces, &

Holomorphic Disks, I

Claude LeBrun
Stony Brook University

Autumn School on Holomorphic Disks
Schloss Rauischholzhausen, November 15, 2018

Joint work with

Joint work with

Lionel Mason
Oxford University

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Zoll Manifolds and Complex Surfaces

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Zoll Manifolds and Complex Surfaces
J. Diff. Geom. 347 (2002) 453–535.

Definition.

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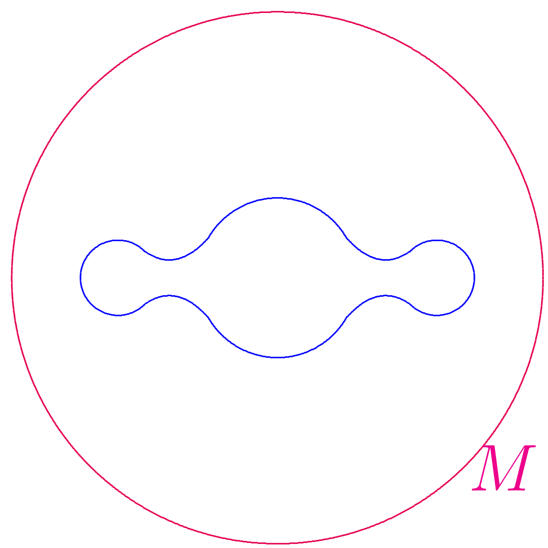

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Equal length: actually automatic on compact M^2.

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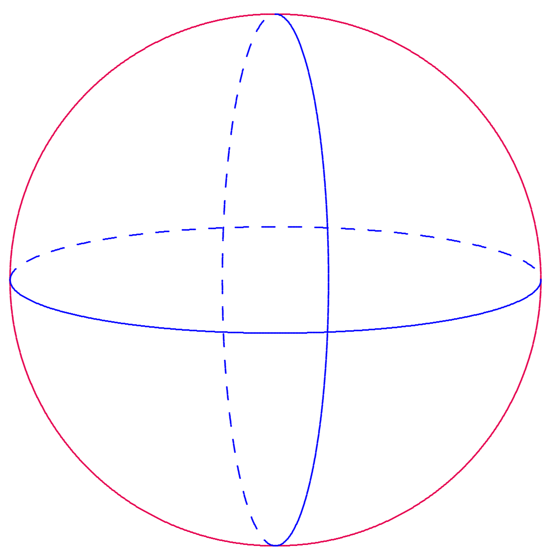

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But Hilbert's conjecture turned out to be false!

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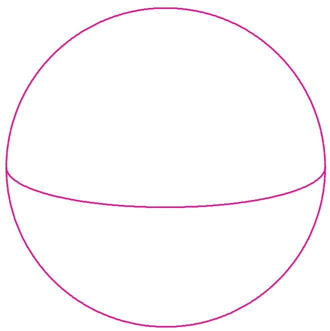

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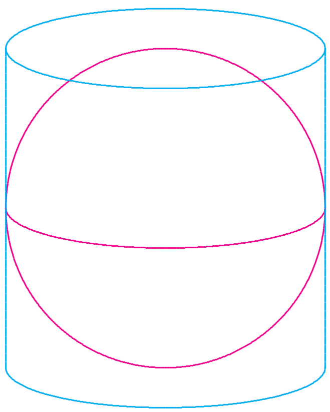

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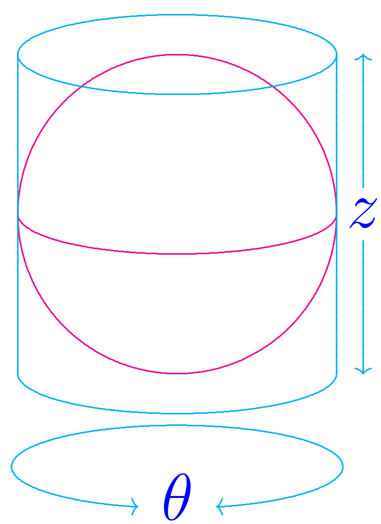

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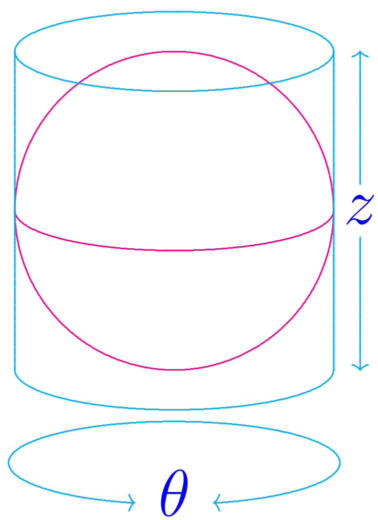

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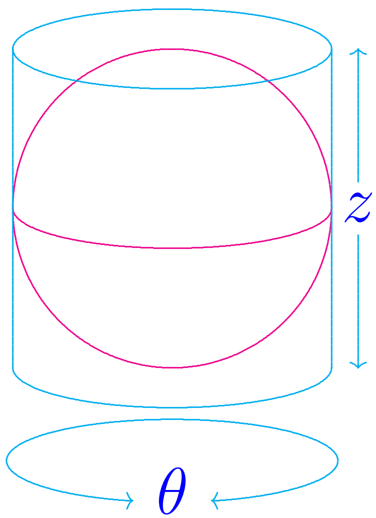

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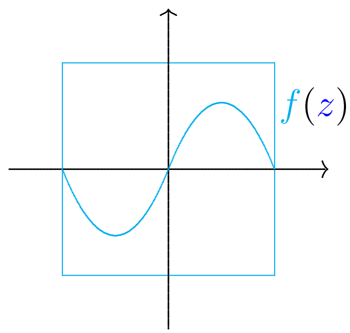

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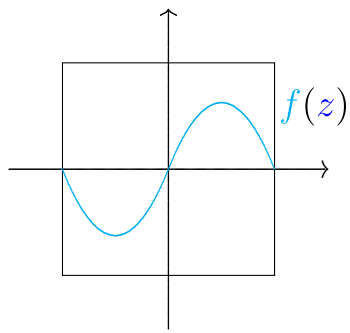

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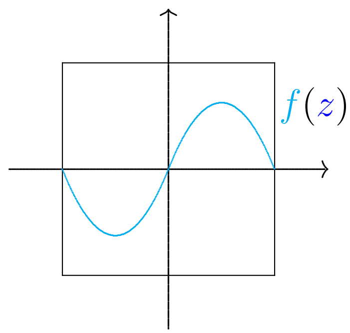

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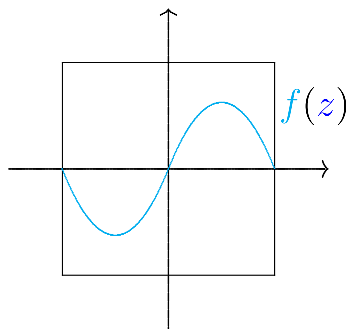

$$\begin{aligned} f : [-1,1] &\to (-1,1), \\ f(-z) &= -f(z), \\ f(1) &= 0. \end{aligned}$$

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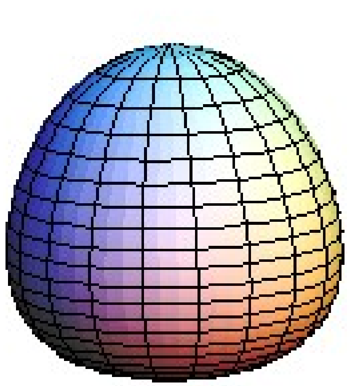

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Embedding in $\mathbb{R}^3$ seems rather unilluminating!

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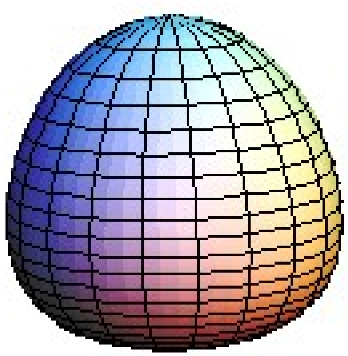

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$\frac{\partial}{\partial \theta}$ is a Killing field, so $z(\theta)$ solves first order ODE.

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Paul Funk (1913): formal power series analysis

$$g = \left[1 + tf + t^2 f_1 + t^3 f_2 + \cdots\right] h$$

$$h = \text{standard metric.}$$

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Victor Guillemin (1976): Proved, using Nash-Moser.

ADVANCES IN MATHEMATICS 22, 85–119 (1976)

The Radon Transform on Zoll Surfaces

VICTOR GUILLEMIN

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

IN MEMORY OF NORMAN LEVINSON

1. INTRODUCTION

The standard metric, ds_0, on S^2 has the property that all its geodesics are closed. Zoll proved (doctoral dissertation, Göttingen, 1901) that there are other smooth metrics on S^2 with this property. His examples are surfaces of revolution (the metric is invariant with respect to the group of rotations about the z-axis); however, it is easy to get examples with this property which are not surfaces of revolution by modifying Zoll's result a bit. Such examples are discussed by Blaschke [1].

We will call a metric on S^2 with the property that the associated geodesic flow, in $T^*S^2 - 0$, is periodic of period 2π a *Zoll surface*.[1] The purpose of this paper is to investigate the existence of Zoll surfaces other than the examples cited above. By the Korn–Lichtenstein theorem every metric on S^2 is conformally equivalent to the standard metric, so we will confine ourselves to looking for Zoll surfaces with metrics of the form $e^{\rho} ds_0$, $\rho \in C^\infty(S^2)$. The set of all ρ for which this metric is a Zoll metric, is a subset of $C^\infty(S^2)$; and a natural question to ask is: What is the tangent space to this set at $\rho = 0$? In other words, for what $\dot\rho \in C^\infty(S^2)$ do there exist Zoll deformations $e^{\rho_t} ds_0$ of the standard metric such that

$$\rho_0 = 0 \quad \text{and} \quad d\rho_t/dt = \dot\rho \quad \text{at} \quad t = 0? \tag{1.1}$$

This problem was first proposed by Hilbert and partly solved by Funk in his doctoral dissertation, written under Hilbert in 1913 [2]. Funk's result is that *a necessary condition for there to exist a Zoll deformation ρ_t satisfying* (1.1) *is that $\dot\rho$ be an odd function, $\dot\rho(-x) = -\dot\rho(x)$ for all $x \in S^2$.* The purpose of this paper is to show that Funk's condition is sufficient

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Actual Blaschke conjecture concerns (M^2, g) s.t. $\forall p \in M$ $\exists!$ conjugate point $p' \neq p \in M$.

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Equivalence was noticed by Reidemeister, who attempted to give a proof based on this observation.

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Leon Green (1963): Correct proof, for C^3 metrics.

ANNALS OF MATHEMATICS
Vol. 78, No. 2, September, 1963
Printed in Japan

AUF WIEDERSEHENSFLÄCHEN*

BY L. W. GREEN

(Received July 26, 1962)

1. A surface on which each point has one and only one conjugate point is called a wiedersehensfläche. (Rigorous definitions will be given below.) In the first edition of his Vorlesungen über Differentialgeometrie [1], Blaschke conjectured that such a surface must have constant curvature, i.e., be isometric with an ordinary euclidean sphere. The second edition [2] contained an appendix by Reidemeister giving a proof of this conjecture by projective methods. In the third edition [3], Blaschke pointed out the error in this proof and described an example of Hjelmslev which showed that this approach cannot work. In particular, there exist non-riemannian metrics on the two sphere whose geodesics behave like those on a wiedersehensfläche.

Other work on this problem was reported by Funk [5] and Gambier [6]. Funk showed that there is no non-trivial analytic one parameter family of wiedersehensflächen containing a sphere of constant curvature.

In 1960 we established a weaker conjecture, analogous to Blaschke's but dealing with focal points instead of conjugate points [7]. Finally, in 1961, we outlined a proof of the original conjecture, under the assumption of positive curvature [8]. Before that announcement appeared in print, however, we had succeeded in finding a simpler proof of the theorem without that assumption.

In § 2 an exact definition of wiedersehensflächen is given and, for the sake of completeness and rigor, some of their geometrical properties described by Blaschke are established. Section 3 is devoted to extending an integral geometric formula of Santaló to cover our slightly more general case. Some analytic properties of the Jacobi equation and the proof of the main theorem are obtained in § 4. In § 5 we generalize the key inequality to n-dimensions.

We should like to express our thanks to H. L. Weinberger and M. Berger for many helpful suggestions.

2. Let M be an orientable two dimensional C^3 surface with a complete C^2 riemannian metric. Denote the unit tangent bundle of M by T and its projection map by ψ. For $e \in T$, let $x(s)$ be the unique geodesic on M, parametrized by arc length, with initial conditions $x(0) = \psi(e)$, $x'(0) = e$.

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Definition. *Let M be a smooth compact manifold. A Zoll conection on M is a torsion-free affine connection ∇ for which each geodesic is a simple closed curve.*

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Definition. *Let M be a smooth compact manifold. A Zoll conection on M is a torsion-free affine connection ∇ for which the image of each maximally-extended geodesic is a simple closed curve.*

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Definition. *A Zoll projective structure* $[\nabla]$ *on* M *is the projective equivalence class of some Zoll connection* ∇.

Theorem A.

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Notice that only remaining case is $M = S^2 \ldots$

Theorem B.

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Based: fixing three chosen points.

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v divergence-free $\Leftrightarrow$ has odd Hamiltonian f on S^2.

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More precise and geometric version tomorrow...

Proofs by twistor methods.

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Poncelet duality:

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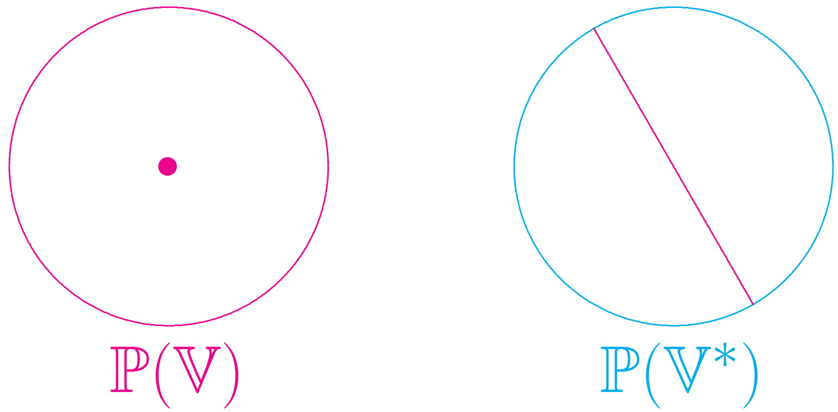

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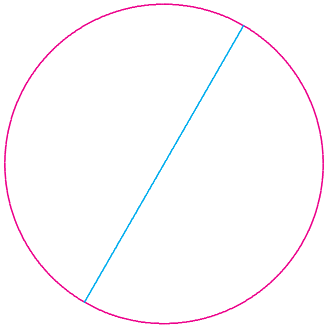

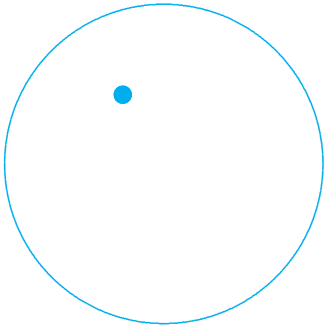

$\mathbb{P}(\mathbb{V})$ $\mathbb{P}(\mathbb{V}^*)$

Proofs by twistor methods.

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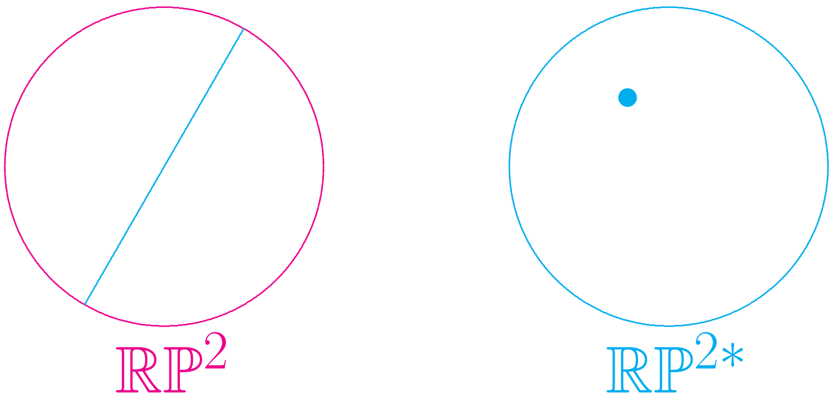

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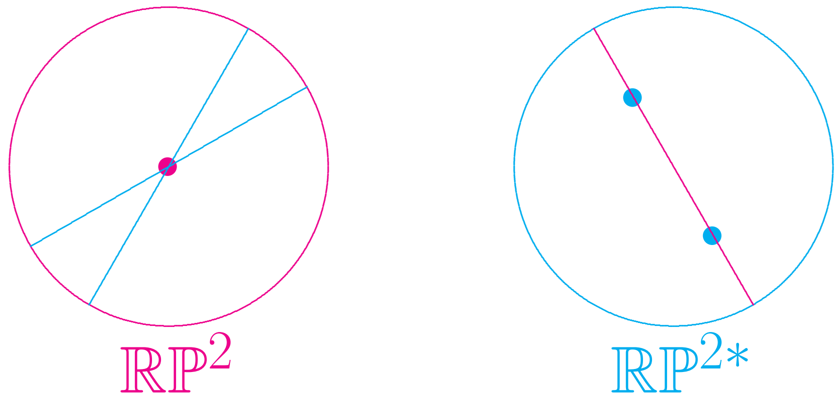

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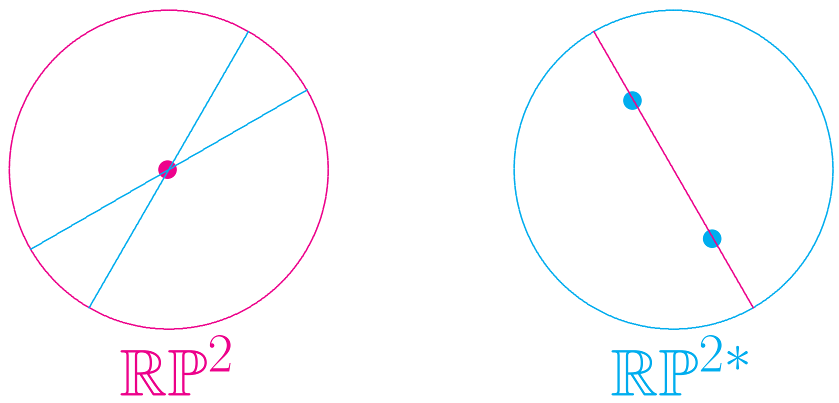
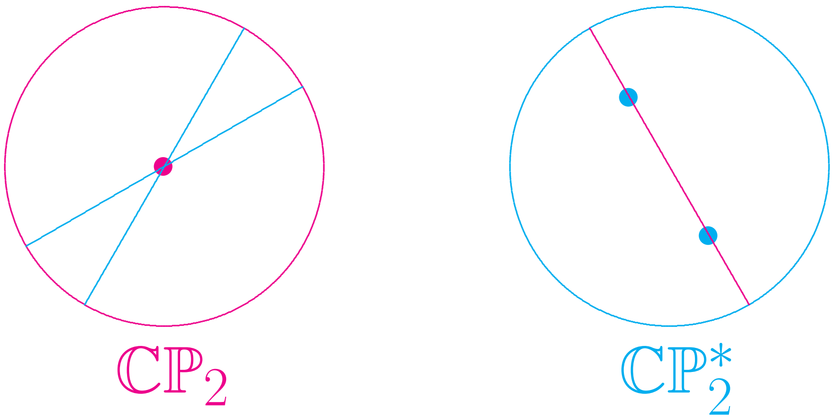


This is where Reidemeister tried and failed!

Proofs by twistor methods.

Poncelet duality: (over $\mathbb{C}$)



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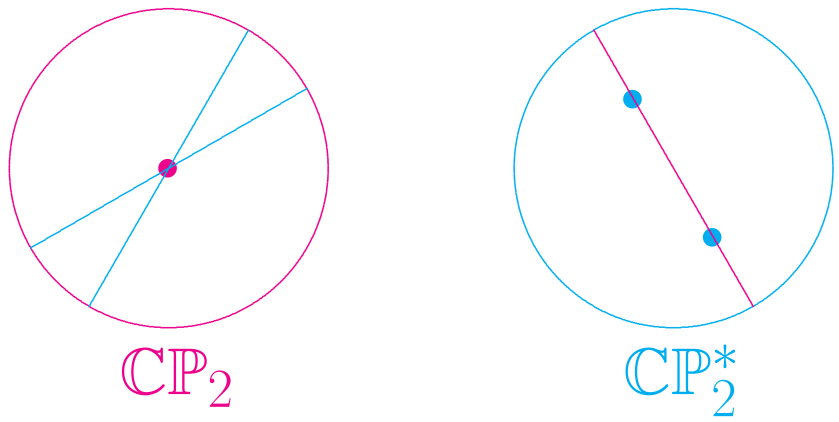
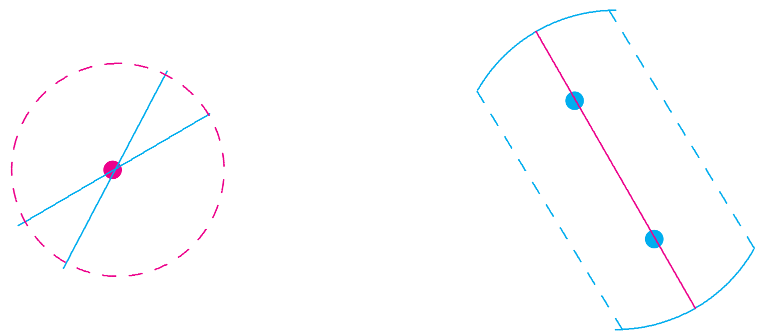


Complex geometry has useful natural rigidity!

Proofs by twistor methods.

Poncelet duality: Localized version



$\mathbb{CP}_2$ $\mathbb{CP}_2^*$

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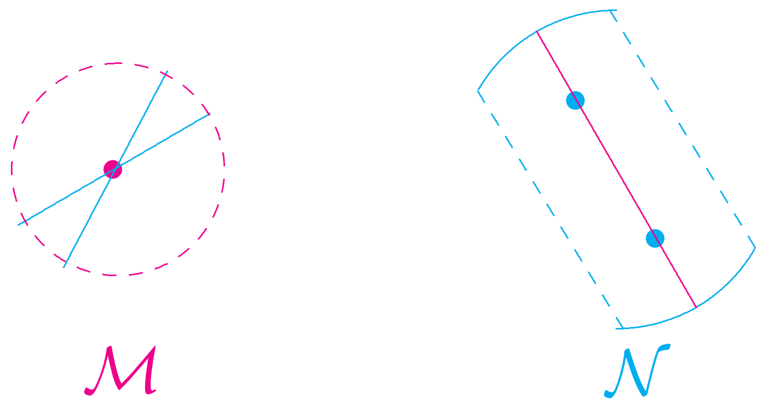
A "traditional" twistor correspondence:



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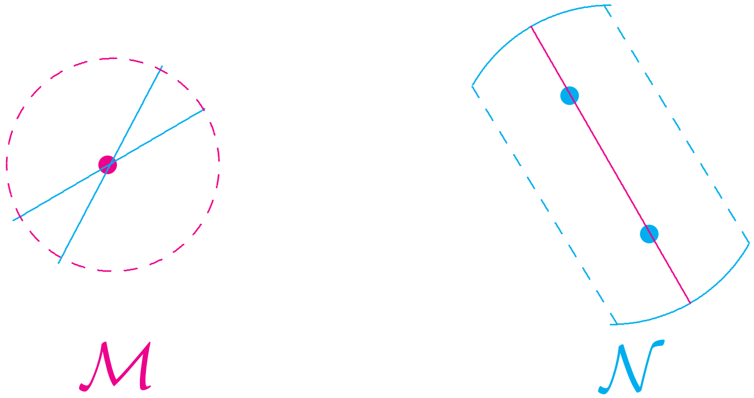

$\mathcal{N}$ complex surface $\supset \mathbb{CP}_1$ of normal bundle $\mathcal{O}(1)$

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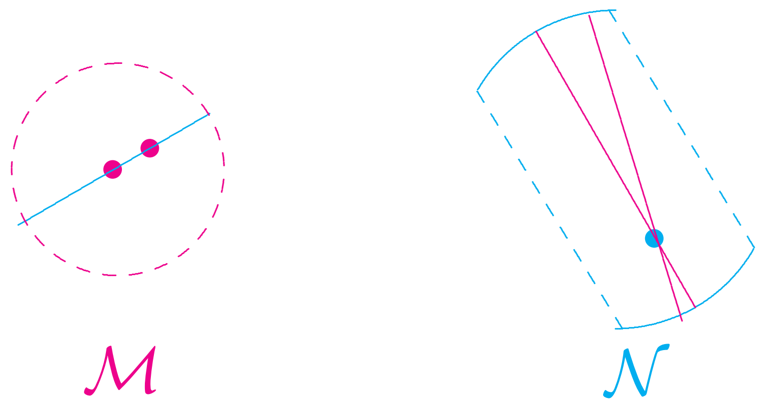

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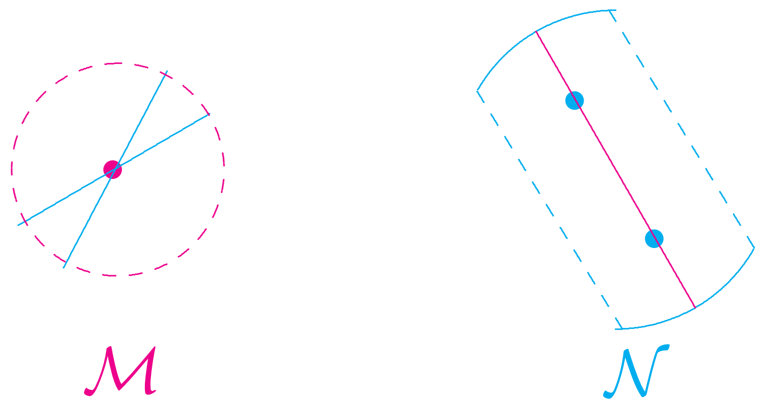

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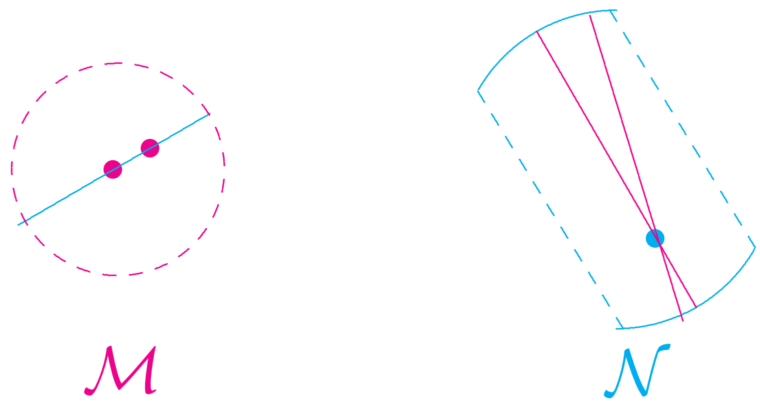

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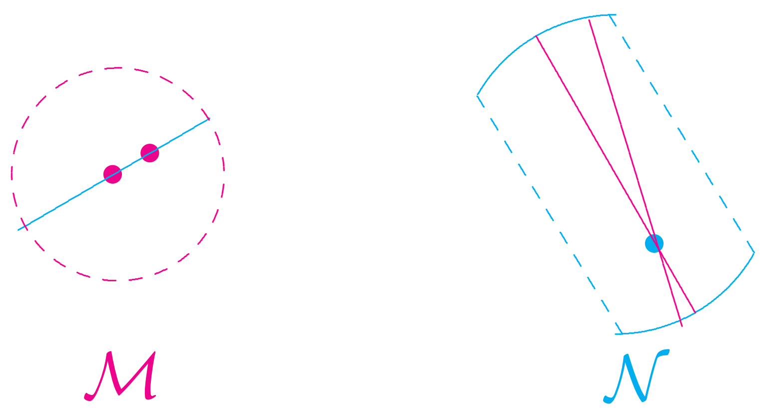

Theorem (Kodaira).

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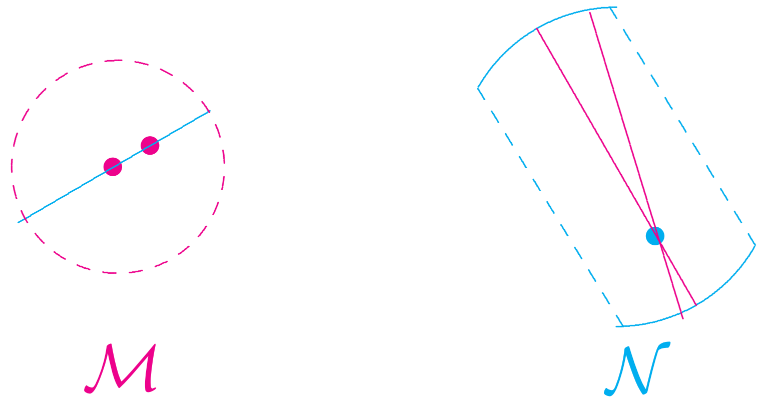

Theorem (Kodaira). *Let* $X \subset Y$ *be a compact complex submanifold*

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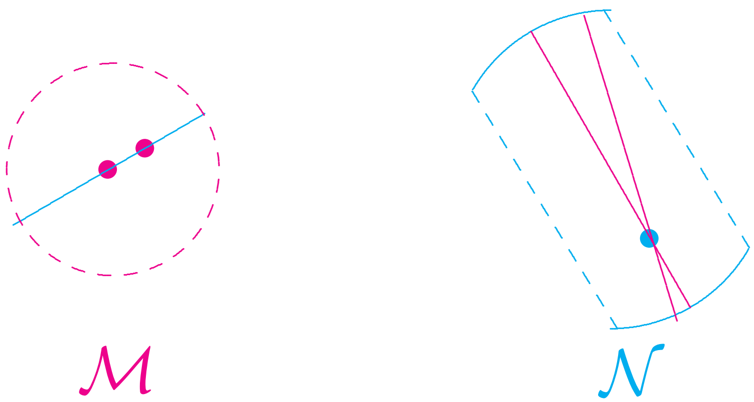

Theorem (Kodaira). *Let* $X \subset Y$ *be a compact complex submanifold, with normal bundle* $\nu = T^{1,0}Y/T^{1,0}X$.

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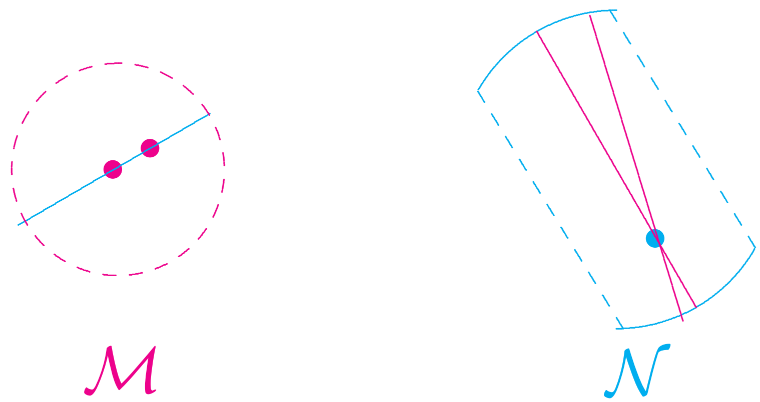

Theorem (Kodaira). *Let $X \subset Y$ be a compact complex submanifold, with normal bundle $\nu = T^{1,0}Y/T^{1,0}X$. If $H^1(X, \mathcal{O}(\nu)) = 0$,*

Proofs by twistor methods.

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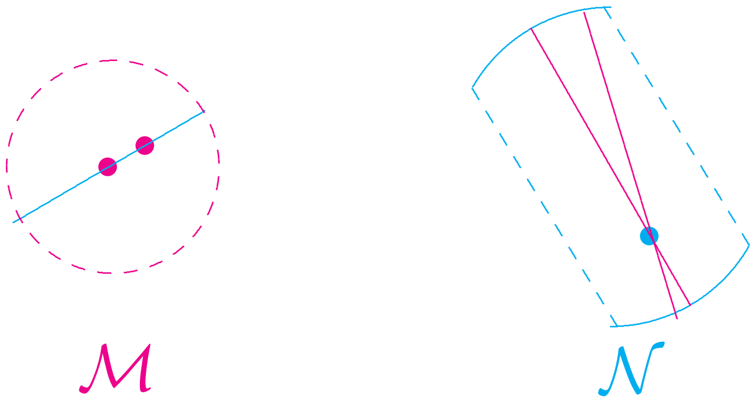

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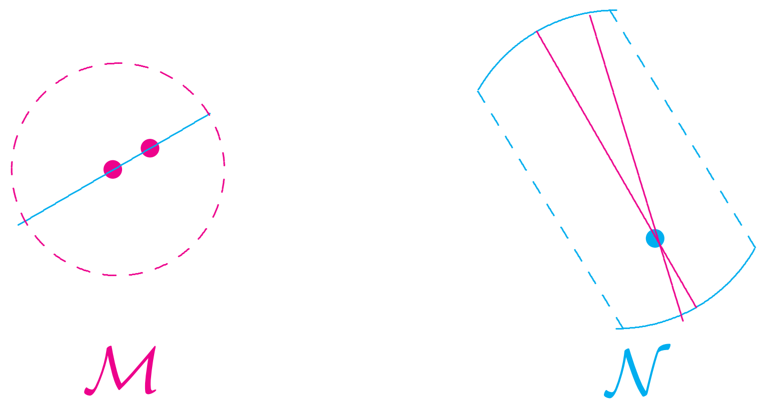

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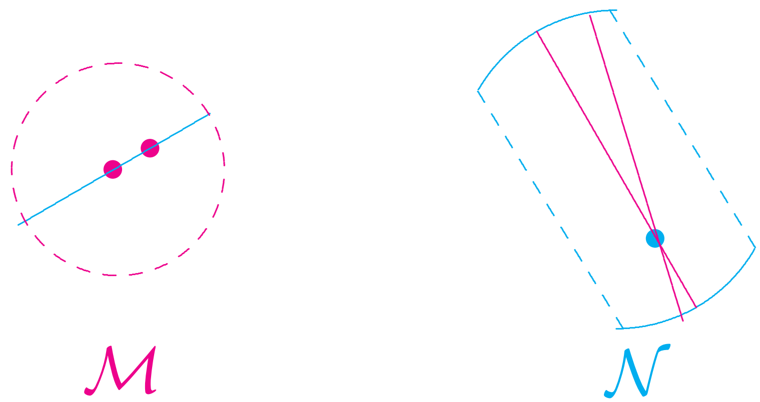

Theorem (L 1980, Hitchin 1982).

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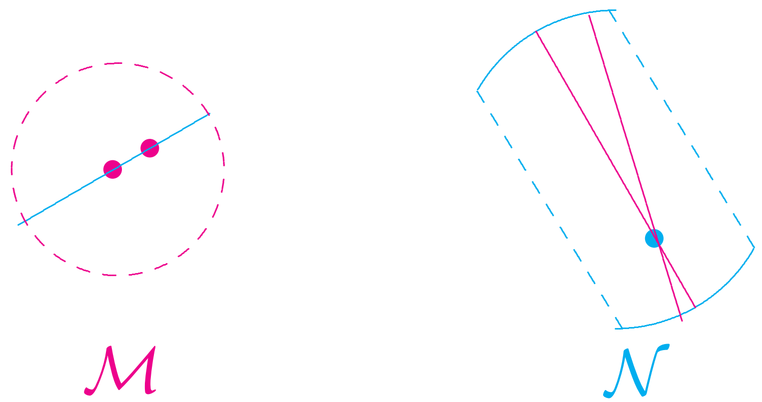

Theorem (L 1980, Hitchin 1982)**.** *Let* $\mathcal{N}$ *be a complex surface*

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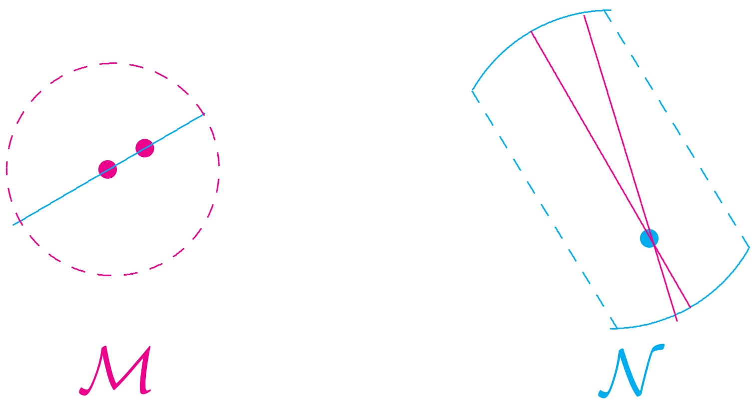

Theorem (L 1980, Hitchin 1982)**.** *Let* $\mathcal{N}$ *be a complex surface that contains a* $\mathbb{CP}_1$

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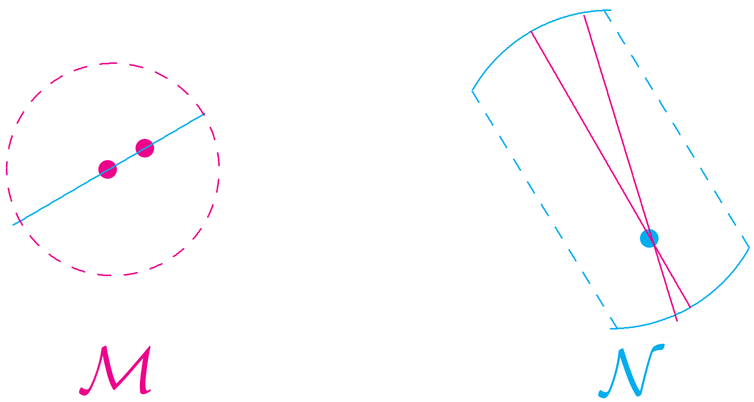

Theorem (L 1980, Hitchin 1982). *Let $\mathcal{N}$ be a complex surface that contains a $\mathbb{CP}_1$ of normal bundle $\mathcal{O}(1)$.*

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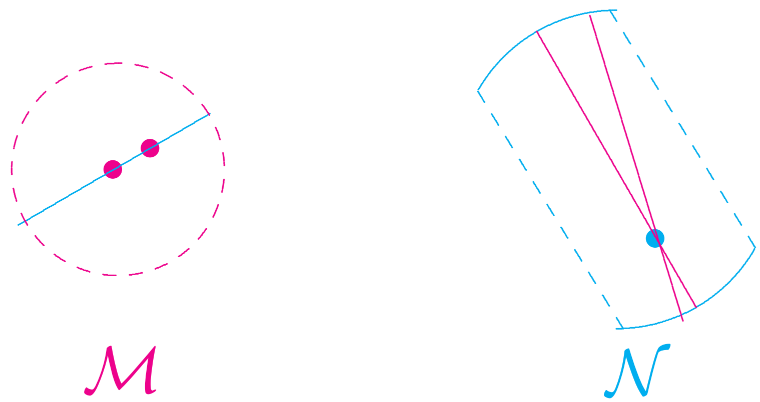

Theorem (L 1980, Hitchin 1982). *Let $\mathcal{N}$ be a complex surface that contains a $\mathbb{CP}_1$ of normal bundle $\mathcal{O}(1)$. Then the moduli space of all such complex curves in $\mathcal{N}$*

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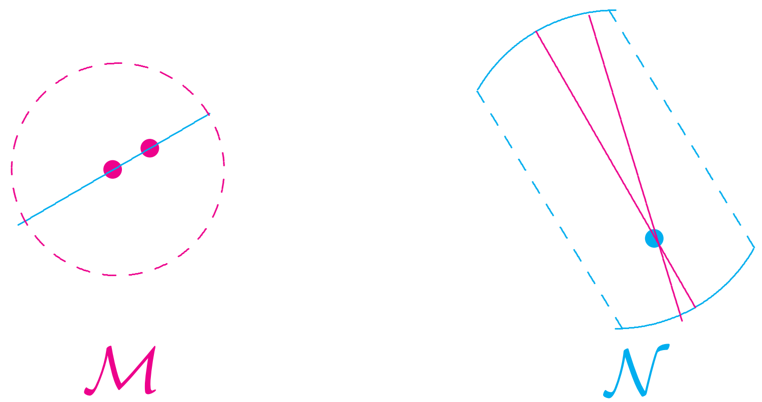
A "traditional" twistor correspondence:



Theorem (L 1980, Hitchin 1982)**.** *Let $\mathcal{N}$ be a complex surface that contains a $\mathbb{CP}_1$ of normal bundle $\mathcal{O}(1)$. Then the moduli space of all such complex curves in $\mathcal{N}$ is a complex surface $\mathcal{M}$,*

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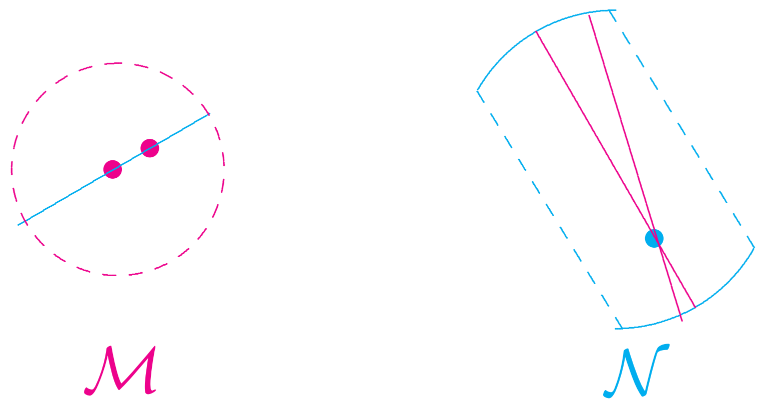

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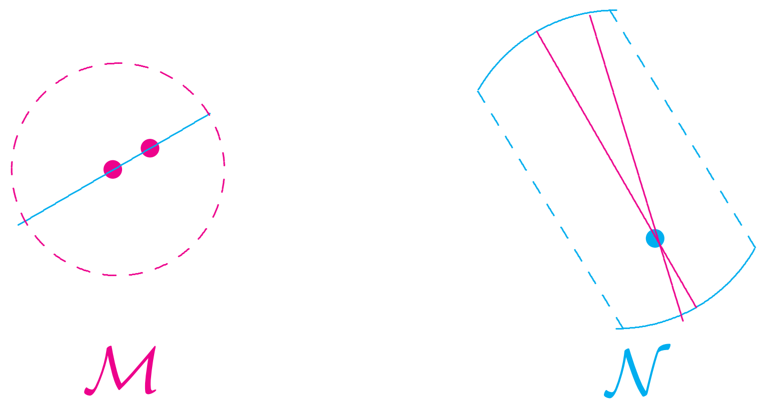

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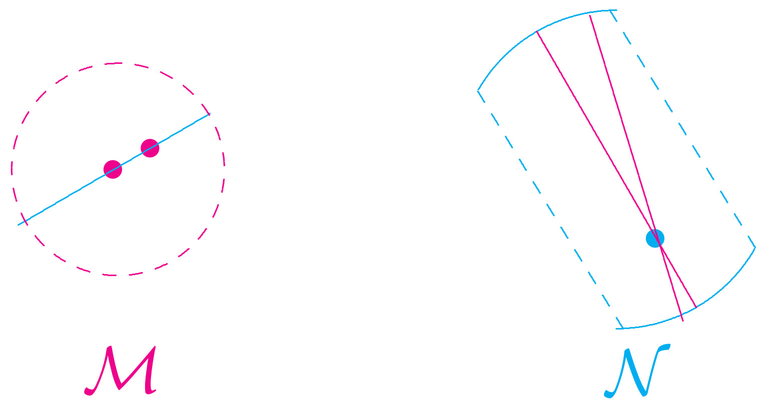
Proofs by twistor methods.

A "traditional" twistor correspondence:



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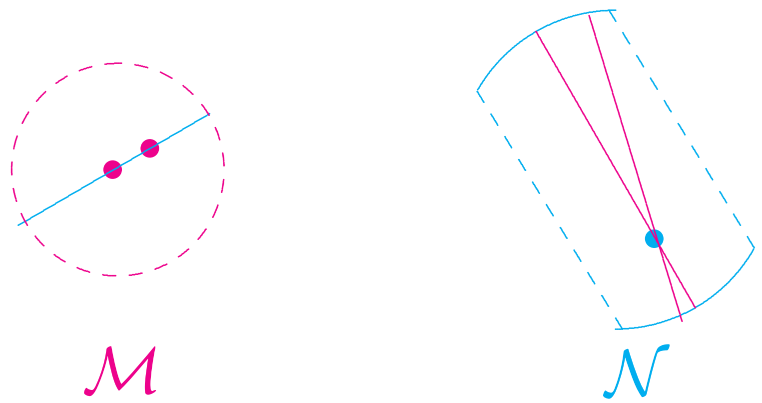

$\mathcal{N}$ = space of complex geodesics in $(\mathcal{M}, [\nabla])$.

$\mathcal{M}$ = space of $\mathbb{CP}_1$'s of self-intersection $+1$ in $\mathcal{N}$.

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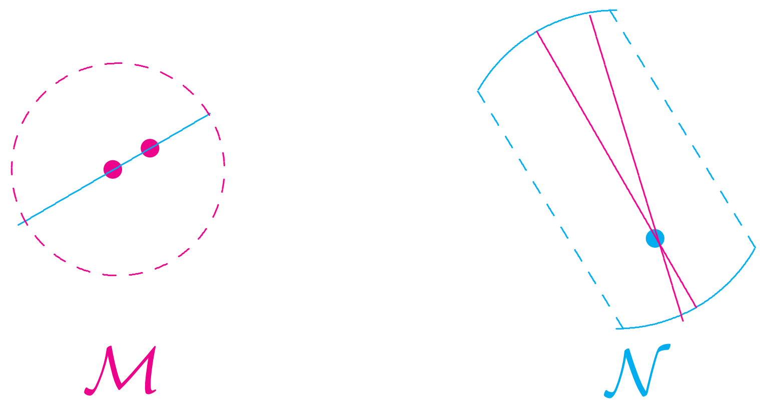

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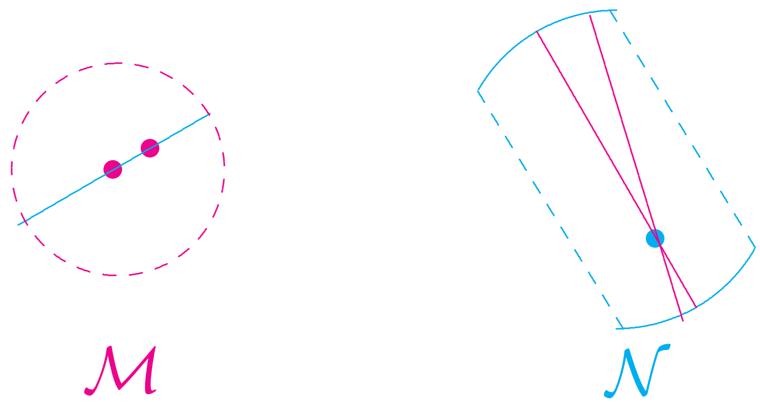

Limitations:

Proofs by twistor methods.

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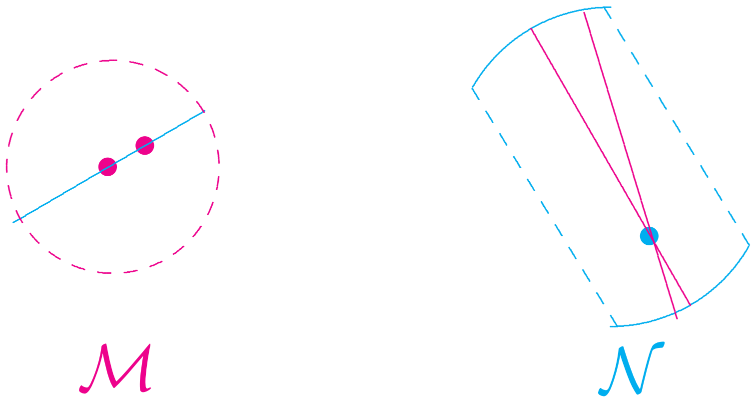

Limitations:

- Doesn't naturally lead to global results.

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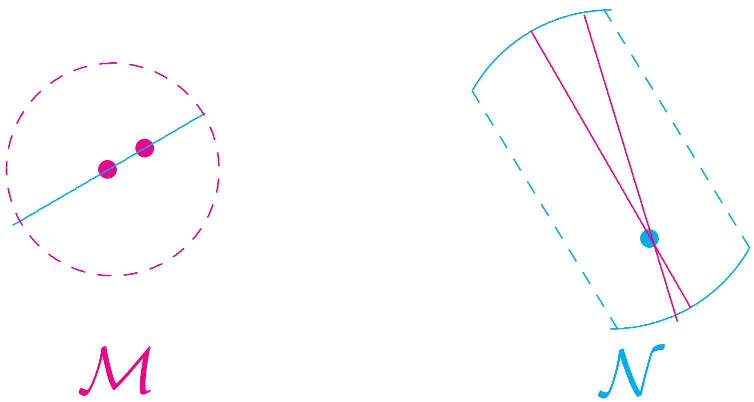

Limitations:

- Doesn't naturally lead to global results.
- Geared to complex-analytic geometry.

Proofs by twistor methods.

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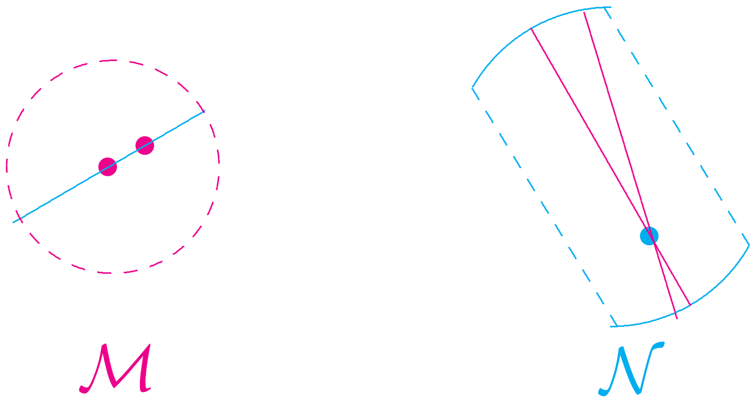

Limitations:

- Doesn't naturally lead to global results.
- Geared to complex-analytic geometry.
- Real-analytic geometry via analytic continuation.

Proofs by twistor methods.

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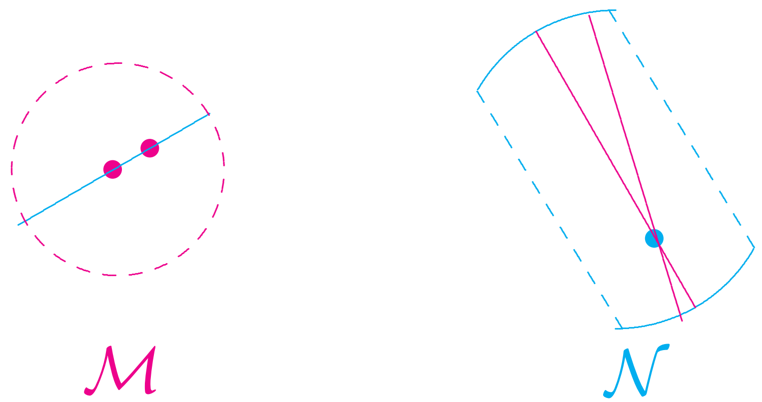

Limitations:

- Doesn't naturally lead to global results.
- Geared to complex-analytic geometry.
- Real-analytic geometry via analytic continuation.
- But doesn't apply to smooth real geometries.

Combine the real and complex pictures?

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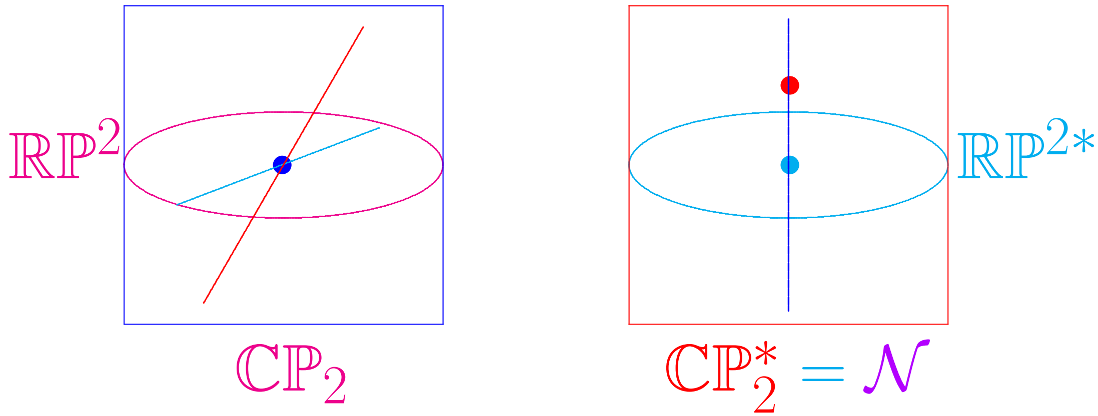

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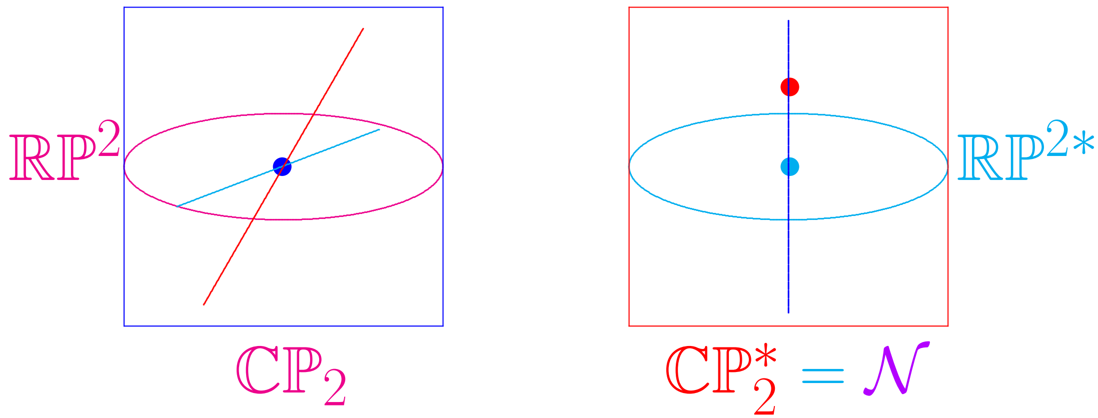

A projective line in $\mathbb{RP}^{2*}$ bounds two disks in $\mathcal{N}$.

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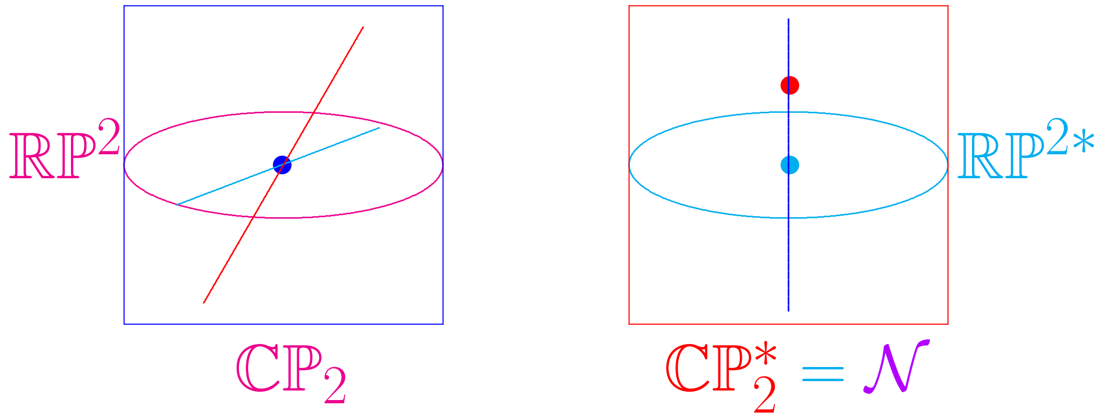

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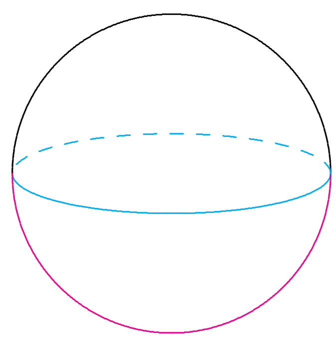

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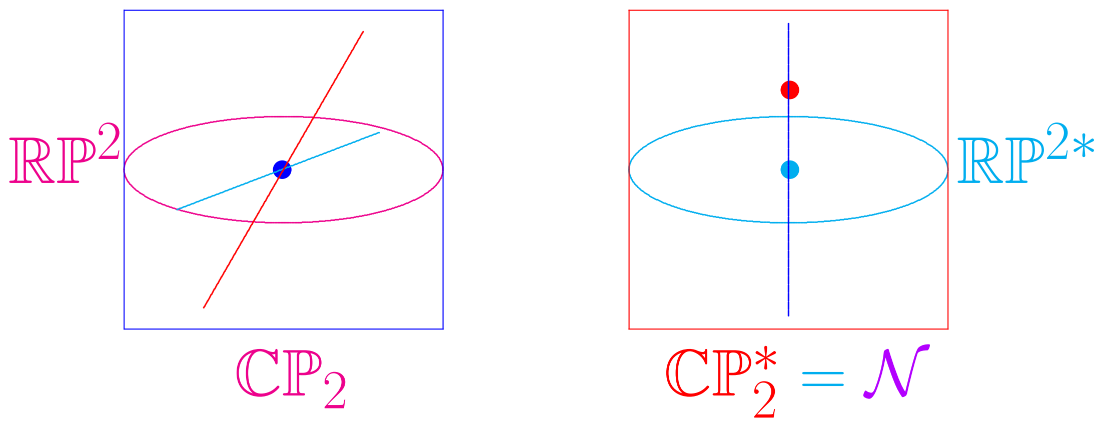

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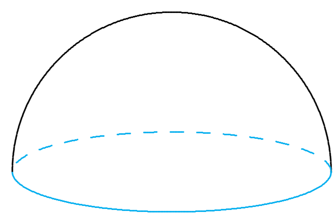

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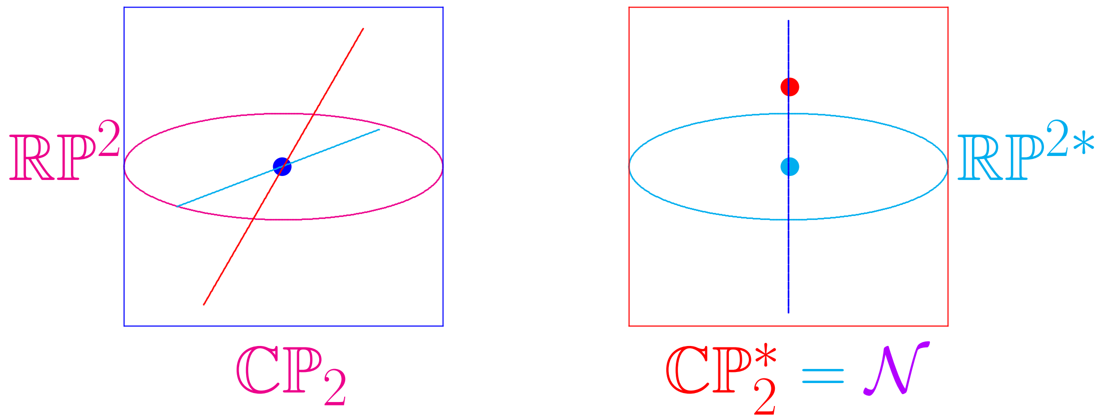

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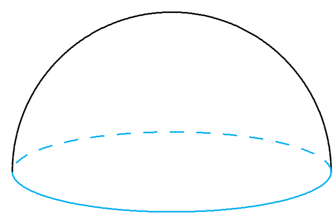
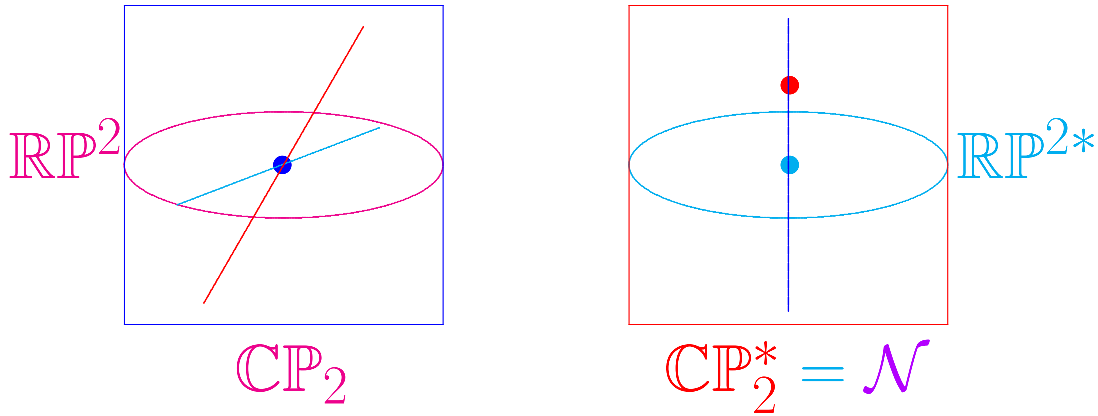


Key: holomorphic disks with boundary on an $\mathbb{RP}^2$.

Combine the real and complex pictures?



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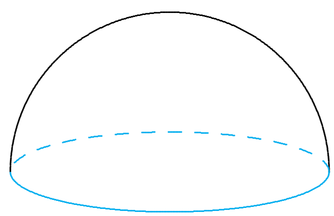

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New kind of twistor correspondence...

End, Part I