On

Hermitian, Einstein

4-Manifolds

Claude LeBrun
Stony Brook University
For Eugenio Calabi
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who discovered the magic link between Einstein manifolds and complex geometry.
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“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
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As punishment ...
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Has same sign as the scalar curvature

$$s = r^j_j = R^{ij}{}_{ij}.$$
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In local complex coordinates $(z^1, \ldots, z^m)$,

$$h = \sum_{j,k=1}^{m} h_{j\bar{k}} \left[ dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$
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where $[h_{j\bar{k}}]$ Hermitian matrix at each point.
If $(M^{2m}, h, J)$ is Hermitian, then
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\[ \omega(\cdot, \cdot) = h(J\cdot, \cdot) \]
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\[\omega = i \sum_{j,k=1}^{m} h_{jk} \ dz^j \wedge d\bar{z}^k\]
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If \(d\omega = 0\), \((M^{2m}, h, J)\) is called Kähler.
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\([\omega] \in H^2(M, \mathbb{R})\) called the Kähler class.
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The 2-form

\(ir(J\cdot, \cdot)\)
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is curvature of canonical line bundle \(K = \Lambda^{m,0}\).
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In local complex coordinates

\[r_{j\bar{k}} = -\frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[h_{\ell\bar{m}}]\]
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Both are actually Hermitian.
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Recall:

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Connected sum \#: 

\[ \includegraphics[width=0.5\textwidth]{connected_sum.png} \]
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\[
\begin{array}{c}
\hline
\end{array}
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\[ \text{Diagram of connected sum} \]
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![Diagram of connected sum](image-url)
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Connected sum \( \# \):

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If \( N \) is a complex surface, may replace \( p \in N \) with \( \mathbb{CP}_1 \).
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$$M \approx N \# \overline{\mathbb{CP}^2}$$
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in which new \( \mathbb{CP}_1 \) has self-intersection \(-1\).
Theorem A. Let $(M^4, J)$ be a compact complex surface, and suppose that $h$ is an Einstein metric on $M$ which is Hermitian with respect to $J$:

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Then either

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- $M \approx \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, and $h$ is a constant times the Page metric; or
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Exceptional cases: \(\mathbb{CP}_2\) blown up at 1 or 2 points.
Theorem B. Let \((M^4, J)\) be a compact complex surface. Then there is an Einstein metric \(h\) on \(M\) which is Hermitian with respect to \(J\) \(\iff\) \(c_1(M)\) “has a sign.”
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More precisely, there is a Hermitian, Einstein metric \(h\) with Einstein constant \(\lambda\) \iff \((M, J)\) carries a Kähler class \([\omega]\) such that

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c_1(M) = \lambda [\omega].
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For fixed \(\lambda \neq 0\), this \(h\) is moreover unique modulo biholomorphisms of \((M, J)\).
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Kähler case: Calabi, Aubin, Yau, Siu, Tian, …
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Warning: when \(h\) is non-Kähler, its relation to \(\omega\) is surprisingly complicated!
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Fano manifolds of complex dimension 2.
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$$M \simeq \begin{cases} 
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\(k \neq 1, 2 \implies \text{admit Kähler-Einstein metrics.}\)
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For fixed $\lambda \neq 0$, this $h$ is moreover unique modulo biholomorphisms of $(M, J)$.

Non-Kähler cases: $\mathbb{CP}^2$ blown up at 1 or 2 points.
Lemma. Let $(M^4, J)$ be a compact complex surface, and suppose that $h$ is an Einstein metric on $M$ which is Hermitian with respect to $J$:

$$h(J\cdot, J\cdot) = h.$$
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Then \((M^4, h, J)\) is conformally Kähler!

In other words,

\[
h = fg
\]

∃ Kähler metric \(g\), smooth function \(f : M \to \mathbb{R}^+\).
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Strictly four-dimensional phenomenon.
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Similarly for \(S^{2n+1} \times S^{2m+1}\).
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Actually, $g$ must be an extremal Kähler metric in sense of Calabi!
Calabi:
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Extremal Kähler metrics = critical points of

\[ g \mapsto \int_{M} s^2 d\mu_g \]
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where \( g = g_\omega \) for \( J \) and \([\omega] \in H^2(M, \mathbb{R})\) fixed.
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Euler-Lagrange equations
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Euler-Lagrange equations \( \iff \)

\( \nabla^{1,0}s \) is a holomorphic vector field.
Calabi:

Extremal Kähler metrics $= \text{critical points of}$

$$g \mapsto \int_M s^2 d\mu_g$$

where $g = g_\omega$ for $J$ and $[\omega] \in H^2(M, \mathbb{R})$ fixed.

Euler-Lagrange equations $\iff$

$$\nabla \nabla s = (\nabla \nabla s)(J\cdot, J\cdot).$$
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X.X. Chen: always minimizers.
Calabi:

Extremal Kähler metrics = critical points of

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where \( g = g_\omega \) for \( J \) and \([\omega] \in H^2(M, \mathbb{R})\) fixed.

Euler-Lagrange equations \( \iff \)

\( J\nabla s \) is a Killing field.

Donaldson/Mabuchi/Chen-Tian:
unique in Kähler class, modulo bihomorphisms.
Lemma. Let \((M^4, J)\) be a compact complex surface, and suppose that \(h\) is an Einstein metric on \(M\) which is Hermitian with respect to \(J\):

\[ h(J\cdot, J\cdot) = h. \]

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What’s so special about dimension four?
Special character of dimension 4:
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On oriented \((M^4, g)\),

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\Lambda^2 = \Lambda^+ \oplus \Lambda^-
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where \(\Lambda^\pm\) are \((\pm 1)\)-eigenspaces of
\[
\star : \Lambda^2 \to \Lambda^2,
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\[
\star^2 = 1.
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\(\Lambda^+\) self-dual 2-forms.
\(\Lambda^-\) anti-self-dual 2-forms.
Riemann curvature of $g$

$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$
Riemann curvature of $g$

\[ \mathcal{R} : \Lambda^2 \rightarrow \Lambda^2 \]

splits into 4 irreducible pieces:

\[ \mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \mathring{r} \\
\mathring{r} & W_- + \frac{s}{12}
\end{pmatrix}. \]
Riemann curvature of $g$

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splits into 4 irreducible pieces:

\[
\begin{array}{c|c|c}
\Lambda^+ & W_+ + \frac{s}{12} & \hat{r} \\
\hline
\Lambda^- & \hat{r} & W_- + \frac{s}{12}
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where

$s = \text{scalar curvature}$

$\mathring{r} = \text{trace-free Ricci curvature}$

$W_+ = \text{self-dual Weyl curvature}$

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where

$s = \text{scalar curvature}$

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$W_+ = \text{self-dual Weyl curvature} \ (\text{conformally invariant})$

$W_- = \text{anti-self-dual Weyl curvature}$
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]
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\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \]
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\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^\perp \]

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\[ W_+ + \frac{s}{12} = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix} \]
Kähler case:

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Notice that \(W_+\) has a repeated eigenvalue.
Lemma. Let \((M^4, J)\) be a compact complex surface, and suppose that \(h\) is an Einstein metric on \(M\) which is Hermitian with respect to \(J\):
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Key step: show \(W_+\) has a repeated eigenvalue.
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Key step: show \(W_+\) has a repeated eigenvalue.

Riemannian analog of Goldberg-Sachs theorem.
Lemma. Let $(M^4, J)$ be a compact complex surface, and suppose that $h$ is an Einstein metric on $M$ which is Hermitian with respect to $J$:

$$h(J., J.) = h.$$ 

Then $(M^4, h, J)$ is conformally Kähler!

Key step: show $W_+$ has a repeated eigenvalue.

Riemannian analog of Goldberg-Sachs theorem.

$$\nabla \cdot W_+ = 0,$$ while $T^{1,0}M$ isotropic & involutive.
**Lemma.** Let $(M^4, J)$ be a compact complex surface, and suppose that $h$ is an Einstein metric on $M$ which is Hermitian with respect to $J$:
\[ h(J\cdot, J\cdot) = h. \]
Then $(M^4, h, J)$ is conformally Kähler!

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When $W_+ \neq 0$, then use Derdziński’s Theorem.
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Key step: show \(W_+\) has a repeated eigenvalue.

Riemannian analog of Goldberg-Sachs theorem.

When \(W_+ \neq 0\), then use Derdziński’s Theorem.

When \(W_+ \equiv 0\), use global results of Boyer et al.
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \mathbb{R}e(\Lambda^{2,0}) \]

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\[ |W_+|^2 = \frac{s^2}{24} \]
The Bach Tensor
The Bach Tensor

Conformally invariant Riemannian functional:

$$\mathcal{W}_+(g) = 2 \int_M |W_+|^2_g \, d\mu_g.$$
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Conformally invariant Riemannian functional:

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1-parameter family of metrics

\[ g_t := g + t\dot{g} + O(t^2) \]

First variation

\[ \left. \frac{d}{dt} \mathcal{W}_+(g_t) \right|_{t=0} = - \int \dot{g}^{ab} B_{ab} \, d\mu_g \]
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\[ B_{ab} := (2\nabla^c \nabla^d + \ddot{\kappa}^{cd})(W_+)_{acbd}. \]

is the Bach tensor of \( g \). Symmetric, trace-free.
The Bach Tensor

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\[ \nabla^a B_{ab} = 0 \]
The Bach Tensor

Conformally invariant Riemannian functional:
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Conformally Einstein \(\implies B = 0\)
Restriction of $\mathcal{W}_+$ to Kähler metrics?

On Kähler metrics,

$$\int |W_+|^2 d\mu = \int \frac{s^2}{24} d\mu$$

so any critical point of restriction must be extremal in sense of Calabi.
Restriction of $\mathcal{W}_+$ to Kähler metrics?

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In fact, for Kähler metrics,

$$B = \frac{1}{12} \left[ 2s r + \text{Hess}_0(s) + 3 J^* \text{Hess}_0(s) \right]$$

where $\text{Hess}_0$ denotes trace-free part of $\nabla \nabla$. 
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**Lemma.** If $g$ is a Kähler metric on a complex surface $(M^4, J)$, the following are equivalent:
Restriction of \( \mathcal{W}_+ \) to Kähler metrics?

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- $\psi = B(J\cdot, \cdot)$ is a closed 2-form;
Restriction of $\mathcal{W}_+$ to Kähler metrics?

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**Lemma.** If $g$ is a Kähler metric on a complex surface $(M^4, J)$, the following are equivalent:

- $g$ is an extremal Kähler metric;
- $B = B(J\cdot, J\cdot)$;
- $\psi = B(J\cdot, \cdot)$ is a closed 2-form;
- $g_t = g + tB$ is Kähler metric for small $t$. 
Restriction of $\mathcal{W}_+$ to Kähler metrics.

Hence if $g$ is extremal Kähler metric,

$$g_t = g + tB$$

is a family of Kähler metrics,
Restriction of $\mathcal{W}_+$ to Kähler metrics.

Hence if $g$ is extremal Kähler metric,

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Restriction of $\mathcal{W}_+$ to Kähler metrics.

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$$\omega_t = \omega + t\psi$$

and first variation is

$$\left. \frac{d}{dt} \mathcal{W}_+(g_t) \right|_{t=0} = \int \dot{g}^{ab} B_{ab} \ d\mu_g$$

$$= - \int |B|^2 \ d\mu_g$$
Restriction of $\mathcal{W}_+$ to Kähler metrics.

Hence if $g$ is extremal Kähler metric,

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$$= - \int |B|^2 \, d\mu_g$$

So the critical metrics of restriction of $\mathcal{W}_+$ to \{Kähler metrics\} are Bach-flat Kähler metrics.
Action Function on Kähler Cone
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For any extremal Kähler \((M^4, g, J)\),
Action Function on Kähler Cone

For any extremal Kähler \((M^4, g, J)\),

\[
\frac{1}{32\pi^2} \int s^2 d\mu_g = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2} \| F[\omega] \|^2
\]
Action Function on Kähler Cone

For any extremal Kähler \((M^4, g, J)\),

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where \(F\) is Futaki invariant.
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=: \mathcal{A}([\omega])
\]

where $\mathcal{F}$ is Futaki invariant.
**Action Function on Kähler Cone**

For any extremal Kähler \((M^4, g, J)\),

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\frac{1}{32\pi^2} \int s^2 d\mu_g = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2} \|F_\omega\|^2
\]

\[=: \mathcal{A}([\omega])\]

where \(\mathcal{F}\) is Futaki invariant.

\(\mathcal{A}\) is function on Kähler cone \(\mathcal{K} \subset H^2(M, \mathbb{R})\).
\( \mathcal{K} \subset H^{1,1}(M, \mathbb{R}) = H^{2}(M, \mathbb{R}) \)

\((M\ \text{Del Pezzo})\)
\[ \mathcal{K} \subset H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R}) \]

(M Del Pezzo)
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Action Function on Kähler Cone

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\]

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Lemma. If \(g\) is a Kähler metric on a compact complex surface \((M^4, J)\), with Kähler class \([\omega]\),
Action Function on Kähler Cone

For any extremal Kähler \((M^4, g, J)\),

\[
\frac{1}{32\pi^2} \int s^2 d\mu_g = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2} \| \mathcal{F}[\omega] \|^2
=: \mathcal{A}([\omega])
\]

where \(\mathcal{F}\) is Futaki invariant.

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**Action Function on Kähler Cone**

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- \(g\) is an extremal Kähler metric; and
- \([\omega]\) is a critical point of \(A : \mathcal{K} \rightarrow \mathbb{R}\).
Lemma. Suppose compact complex surface $(M^4, J)$ admits a Hermitian $h$ which is Einstein, but not Kähler. Then $(M^4, J)$ is a Del Pezzo surface.
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\[
0 = 6s^{-1}B = \dot{r} + 2s^{-1}\text{Hess}_0(s)
\]
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\[
\rho + 2i\partial\bar{\partial}\log s > 0.
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Lemma. Suppose compact complex surface \((M^4, J)\) admits a Hermitian \(h\) which is Einstein, but not Kähler. Then \((M^4, J)\) is a Del Pezzo surface.

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Lemma. Conversely, any Hermitian, Einstein metric on a Del Pezzo surface arises in this way.
Theorem. Let \((M^4, J)\) be a Del Pezzo surface. Then, up to automorphisms and rescaling, there is a unique Bach-flat Kähler metric \(g\) on \(M\). This metric is characterized by the fact that it minimizes the Calabi functional

\[
   C = \int_M s^2 d\mu
\]

among all Kähler metrics on \((M^4, J)\).
**Theorem.** Let \((M^4, J)\) be a Del Pezzo surface. Then, up to automorphisms and rescaling, there is a unique Bach-flat Kähler metric \(g\) on \(M\). This metric is characterized by the fact that it minimizes the Calabi functional

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Hermitian, Einstein metric then given by

\[ h = s^{-2} g \]

and uniqueness Theorem A follows.
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among all Kähler metrics on $(M^4, J)$.

Only three cases are non-trivial:

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad k = 1, 2, 3.$$
The non-trivial cases are toric, and the action $A$ can be directly computed from moment polygon.
The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon. Formula involves barycenters, moments of inertia.

\[
\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2} \left( \frac{1}{|P|} + \vec{\Omega} \cdot \Pi^{-1} \vec{\Omega} \right)
\]
To prove Theorem, show that

\[ A : \mathcal{K} \to \mathbb{R} \]

has unique critical point for relevant \( M \).
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Here \( \mathcal{K} = \mathcal{K}/\mathbb{R}^+ \).
To prove Theorem, show that

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Here \( \tilde{\mathcal{K}} = \mathcal{K}/\mathbb{R}^+ \).

\( \mathcal{A} \) is explicit rational function —
3[3 + 28\gamma + 96\gamma^2 + 168\gamma^3 + 164\gamma^4 + 80\gamma^5 + 16\beta^6 (1 + \gamma)^3 + 16\beta^6 (1 + \beta + \gamma)^3 + 16\beta^5 (5 + 24\gamma + 43\gamma^2 + 37\gamma^3 + 15\gamma^4 + 2\gamma^5) + 4\beta^4 (41 + 228\gamma + 478\gamma^2 + 496\gamma^3 + 263\gamma^4 + 60\gamma^5 + 4\gamma^6) + 8\beta^3 (21 + 135\gamma + 326\gamma^2 + 392\gamma^3 + 248\gamma^4 + 74\gamma^5 + 8\gamma^6)] + 4\beta^2 (7 + 58\gamma + 176\gamma^2 + 270\gamma^3 + 228\gamma^4 + 96\gamma^5 + 16\gamma^6) + 4\beta^2 (24 + 176\gamma + 479\gamma^2 + 452\gamma^3 + 478\gamma^4 + 172\gamma^5 + 245\gamma^6) + 16\beta^5 (5 + 3\beta^5 + 24\gamma + 43\gamma^2 + 37\gamma^3 + 15\gamma^4 + 2\gamma^5 + \beta^2 (15 + 14\gamma) + \beta^3 (37 + 70\gamma + 30\gamma^2) + \beta^2 (43 + 123\gamma + 108\gamma^2 + 30\gamma^3) + \beta (24 + 92\gamma + 123\gamma^2 + 70\gamma^3) + 14\gamma^4)] + 4\beta^4 (41 + 43\gamma^6 + 228\gamma + 478\gamma^2 + 496\gamma^3 + 263\gamma^4 + 60\gamma^5 + 4\gamma^6) + 16\beta^4 (60 + 56\gamma) + \beta^2 (263 + 476\gamma + 106\gamma^2) + 8\beta^3 (62 + 169\gamma + 139\gamma^2 + 35\gamma^3) + 2\beta^2 (239 + 876\gamma + 1089\gamma^2 + 556\gamma^3 + 98\gamma^4) + 4\beta (57 + 263\gamma + 438\gamma^2 + 338\gamma^3 + 119\gamma^4 + 14\gamma^5)] + 8\alpha^3 (21 + 135\gamma + 326\gamma^2 + 392\gamma^3 + 248\gamma^4 + 74\gamma^5 + 8\gamma^6) + 8\beta^6 (1 + \gamma) + 2\beta^5 (37 + 70\gamma + 30\gamma^2) + 4\beta^4 (62 + 169\gamma + 139\gamma^2 + 35\gamma^3) + 4\beta^3 (98 + 353\gamma + 428\gamma^2 + 210\gamma^3 + 35\gamma^4) + 2\beta^2 (163 + 735\gamma + 1179\gamma^2 + 856\gamma^3 + 278\gamma^4 + 30\gamma^5) + \beta (135 + 736\gamma + 1470\gamma^2 + 1412\gamma^3 + 676\gamma^4 + 140\gamma^5 + 8\gamma^6)] + 4\alpha (7 + 58\gamma + 176\gamma^2 + 270\gamma^3 + 228\gamma^4 + 96\gamma^5 + 16\gamma^6 + 16\beta^4 (1 + \gamma)^3 + 4\beta^5 (24 + 92\gamma + 123\gamma^2 + 70\gamma^3 + 14\gamma^4) + 4\beta^4 (57 + 263\gamma + 438\gamma^2 + 338\gamma^3 + 119\gamma^4 + 14\gamma^5) + 2\beta^3 (135 + 736\gamma + 1470\gamma^2 + 1412\gamma^3 + 676\gamma^4 + 140\gamma^5 + 8\gamma^6)] + 4\beta^2 (44 + 278\gamma + 645\gamma^2 + 735\gamma^3 + 438\gamma^4 + 123\gamma^5 + 12\gamma^6) + 2\beta (29 + 216\gamma + 556\gamma^2 + 736\gamma^3 + 526\gamma^4 + 184\gamma^5 + 24\gamma^6) + 4\beta^2 (24 + 176\gamma + 479\gamma^2 + 652\gamma^3 + 478\gamma^4 + 172\gamma^5 + 24\gamma^6 (1 + \gamma)^2 + 4\beta^5 (43 + 123\gamma + 108\gamma^2 + 30\gamma^3) + 2\beta^4 (239 + 876\gamma + 1089\gamma^2 + 556\gamma^3 + 98\gamma^4) + 4\beta^3 (163 + 735\gamma + 1179\gamma^2 + 856\gamma^3 + 278\gamma^4 + 30\gamma^5) + 4\beta^2 (44 + 278\gamma + 645\gamma^2 + 735\gamma^3 + 438\gamma^4 + 123\gamma^5 + 12\gamma^6) + \beta^2 (479 + 258\gamma + 505\gamma^2 + 471\gamma^3 + 217\gamma^4 + 432\gamma^5 + 24\gamma^6)]}} + [1 + 10\gamma + 36\gamma^2 + 64\gamma^3 + 60\gamma^4 + 24\gamma^5 + 24\gamma^6 (1 + \gamma)^5 + 24\gamma^6 (1 + \beta + \gamma)^5 + 12\gamma^6 (1 + \gamma)^2 (5 + 20\gamma + 23\gamma^2 + 10\gamma^3) + 16\beta^3 (4 + 28\gamma + 72\gamma^2 + 90\gamma^3 + 57\gamma^4 + 15\gamma^5) + 12\beta^3 (3 + 24\gamma + 69\gamma^2 + 96\gamma^3 + 68\gamma^4 + 20\gamma^5) + 2\beta (5 + 45\gamma + 144\gamma^2 + 224\gamma^3 + 180\gamma^4 + 60\gamma^5) + 12\beta^2 (1 + \beta + \gamma)^2 (5 + 20\gamma + 23\gamma^2 + 10\gamma^3 + 10\beta^2 (1 + \gamma) + \beta^2 (23 + 46\gamma + 16\gamma^2) + 2\beta (18 + 30\gamma + 23\gamma^2 + 5\gamma^3)) + 16\beta^3 (4 + 28\gamma + 72\gamma^2 + 90\gamma^3 + 57\gamma^4 + 15\beta^3 (1 + \gamma) + 3\beta^3 (19 + 57\gamma + 50\gamma^2 + 13\gamma^3) + 3\beta^3 (30 + 120\gamma + 155\gamma^2 + 78\gamma^3 + 15\gamma^4) + 3\beta^3 (24 + 120\gamma + 206\gamma^2 + 155\gamma^3 + 50\gamma^4 + 5\gamma^5)] + \beta (28 + 168\gamma + 360\gamma^2 + 360\gamma^3 + 171\gamma^4 + 30\gamma^5)] + 12\beta^2 (3 + 24\gamma + 69\gamma^2 + 96\gamma^3 + 68\gamma^4 + 20\gamma^5 + 24\gamma^6 (1 + \gamma)^3 + \beta^3 (88 + 272\gamma + 366\gamma^2 + 200\gamma^3 + 36\gamma^4) + 4\beta^3 (24 + 120\gamma + 206\gamma^2 + 155\gamma^3 + 50\gamma^4 + 5\gamma^5) + 2\beta (12 + 84\gamma + 207\gamma^2 + 240\gamma^3 + 136\gamma^4 + 30\gamma^5) + \beta^2 (69 + 414\gamma + 864\gamma^2 + 824\gamma^3 + 366\gamma^4 + 60\gamma^5)] + 2\beta (5 + 45\gamma + 144\gamma^2 + 224\gamma^3 + 180\gamma^4 + 60\gamma^5 + 60\beta^5 (1 + \gamma)^4 + 12\beta^4 (15 + 75\gamma + 136\gamma^2 + 114\gamma^3 + 43\gamma^4 + 5\gamma^5) + 12\beta^2 (12 + 84\gamma + 207\gamma^2 + 240\gamma^3 + 136\gamma^4 + 30\gamma^5) + 8\beta^3 (28 + 168\gamma + 360\gamma^2 + 360\gamma^3 + 171\gamma^4 + 30\gamma^5) + 3\beta (15 + 120\gamma + 336\gamma^2 + 448\gamma^3 + 300\gamma^4 + 80\gamma^5)]}
To prove Theorem, show that

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has unique critical point for relevant $M$.

Here $\mathcal{K} = \mathcal{K}/\mathbb{R}^+$. 

$A$ is explicit rational function — but quite complicated!
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Proof proceeds by showing critical point invariant under certain discrete automorphisms of $M$. 
To prove Theorem, show that

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Proof proceeds by showing critical point invariant under certain discrete automorphisms of \( M \).

Done by showing \( \mathcal{A} \) convex on appropriate lines.
To prove Theorem, show that

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Proof proceeds by showing critical point invariant under certain discrete automorphisms of \( M \).

Done by showing \( \mathcal{A} \) convex on appropriate lines.

Final step then just calculus in one variable. . .
To prove Theorem, show that

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$\mathcal{A}$ is explicit rational function — but quite complicated!

Proof proceeds by showing critical point invariant under certain discrete automorphisms of $M$.

Done by showing $\mathcal{A}$ convex on appropriate lines.

Similar calculations also led to new existence proof...
Theorem C. There is a Kähler metric $g$ on $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ which is conformal to an Einstein metric.
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$$[0, 1) \ni t \mapsto g_t$$
Theorem C. There is a Kähler metric \( g \) on \( \mathbb{CP}_2 \# 2 \bar{\mathbb{CP}}_2 \) which is conformal to an Einstein metric. Moreover, there is a 1-parameter family 

\[
[0, 1) \ni t \mapsto g_t
\]

of extremal Kähler metrics on \( \mathbb{CP}_2 \# 3 \bar{\mathbb{CP}}_2 \) s.t.
Theorem C. There is a Kähler metric $g$ on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ which is conformal to an Einstein metric. Moreover, there is a 1-parameter family $[0, 1) \ni t \mapsto g_t$ of extremal Kähler metrics on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ s.t.

- $g_0$ is Kähler-Einstein, and such that
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- $g_0$ is Kähler-Einstein, and such that
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This reconstructs Chen-LeBrun-Weber metric.
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• $g_0$ is Kähler-Einstein, and such that
• $g_{t_j} \to g$ in the Gromov-Hausdorff sense for some $t_j \to 1$.

This reconstructs Chen-LeBrun-Weber metric.

Could also reconstruct Page metric this way...
Ingredients:
Ingredients:

- Continuity method
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\[ \Omega_t = (1 - t)c_1 + t\Omega \]
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- LeBrun-Simanca
Ingredients:

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- LeBrun-Simanca
  - Inverse function theorem \( \Rightarrow \) openness.
Ingredients:

- **Continuity method**
  \[ \Omega_t = (1 - t)c_1 + t\Omega \]

- **LeBrun-Simanca**
  - Inverse function theorem \( \Rightarrow \) openness.

- **Chen-Weber**
Ingredients:

- Continuity method
  
  \[ \Omega_t = (1 - t) c_1 + t \Omega \]

- LeBrun-Simanca
  
  - Inverse function theorem \implies openness.

- Chen-Weber
  
  - Gromov-Hausdorff convergence
Ingredients:

- Continuity method
  \[ \Omega_t = (1 - t)c_1 + t\Omega \]
- LeBrun-Simanca
  - Inverse function theorem ⇒ openness.
- Chen-Weber
  - Gromov-Hausdorff convergence...
Ingredients:

- Continuity method
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- Continuity method
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  – Gromov-Hausdorff convergence . . .

- Sobolev Control
Ingredients:

- Continuity method

\[ \Omega_t = (1 - t)c_1 + t\Omega \]

- LeBrun-Simanca
  - Inverse function theorem \( \Rightarrow \) openness.

- Chen-Weber
  - Gromov-Hausdorff convergence

- Sobolev Control
  - Yamabe trick + Gauss-Bonnet
Ingredients:

- Continuity method
  \[ \Omega_t = (1 - t)c_1 + t\Omega \]

- LeBrun-Simanca
  - Inverse function theorem \(\Rightarrow\) openness.

- Chen-Weber
  - Gromov-Hausdorff convergence . . .

- Sobolev Control
  - Yamabe trick + Gauss-Bonnet . . .

- Control bubbling
Ingredients:

- Continuity method
  \[ \Omega_t = (1 - t)c_1 + t\Omega \]

- LeBrun-Simanca
  - Inverse function theorem \( \Rightarrow \) openness.

- Chen-Weber
  - Gromov-Hausdorff convergence . . .

- Sobolev Control
  - Yamabe trick + Gauss-Bonnet . . .

- Control bubbling
  - Toric geometry
Any bubble must be toric, scalar-flat Kähler, ALE:
Any bubble must be toric, scalar-flat Kähler, ALE:

Must contain a holomorphic 2-sphere $S$ with

$$[S] \cdot [S] < 0.$$
Ingredients:

- Continuity method

\[ \Omega_t = (1 - t)c_1 + t\Omega \]

- LeBrun-Simanca
  - Inverse function theorem \( \Rightarrow \) openness.

- Chen-Weber
  - Gromov-Hausdorff convergence…

- Sobolev Control
  - Yamabe trick + Gauss-Bonnet…

- Control bubbling
  - Toric geometry
  - Symplectic 2-spheres \( \leadsto \) Lagrangian 2-spheres
If bubbling occurs as $t_j \nearrow t_\infty$, 
If bubbling occurs as $t_j \nearrow t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$: 
If bubbling occurs as $t_j \nearrow t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$:

$$\Omega_{t_j} \cdot [S] > 0, \quad \Omega_{t_\infty} \cdot [S] = 0.$$
If bubbling occurs as $t_j \nearrow t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$:

$$\Omega_{t_j} \cdot [S] > 0, \quad \Omega_{t_\infty} \cdot [S] = 0.$$  

Since

$$\Omega_t = (1 - t)c_1 + t\Omega$$
If bubbling occurs as $t_j \nearrow t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$:

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Since

$$\Omega_t = (1 - t)c_1 + t\Omega$$

this implies

$$c_1 \cdot [S] > 0.$$
If bubbling occurs as \( t_j \nearrow t_\infty \), \((M, \Omega_{t_j})\) contains symplectic 2-sphere \( S \) with area \( \rightarrow 0 \):

\[
\Omega_{t_j} \cdot [S] > 0, \quad \Omega_{t_\infty} \cdot [S] = 0.
\]

Since

\[
\Omega_t = (1 - t)c_1 + t\Omega
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this implies

\[
c_1 \cdot [S] > 0.
\]

Adjunction:

\[
2 + [S] \cdot [S] > 0.
\]
If bubbling occurs as $t_j \nearrow t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$:

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this implies

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Adjunction:

$$2 + [S] \cdot [S] > 0.$$ 

But

$$[S] \cdot [S] < 0.$$
If bubbling occurs as $t_j \nearrow t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$:

$$\Omega_{t_j} \cdot [S] > 0, \quad \Omega_{t_\infty} \cdot [S] = 0.$$ 

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Adjunction:

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But

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So

$$[S] \cdot [S] = -1.$$
If bubbling occurs as $t_j \searrow t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$:

$$\Omega_{t_j} \cdot [S] > 0, \quad \Omega_{t_\infty} \cdot [S] = 0.$$ 

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Adjunction:

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But

$$[S] \cdot [S] < 0.$$ 

So

$$[S] \cdot [S] = -1.$$ 

But every such class in $M$ represented by a holomorphic curve!
If bubbling occurs as $t_j \to t_\infty$, $(M, \Omega_{t_j})$ contains symplectic 2-sphere $S$ with area $\to 0$:

$$\Omega_{t_j} \cdot [S] > 0, \quad \Omega_{t_\infty} \cdot [S] = 0.$$ 

Since

$$\Omega_t = (1 - t)c_1 + t\Omega$$

this implies

$$c_1 \cdot [S] > 0.$$ 

Adjunction:

$$2 + [S] \cdot [S] > 0.$$ 

But

$$[S] \cdot [S] < 0.$$ 

So

$$[S] \cdot [S] = -1.$$ 

But every such class in $M$ represented by a holomorphic curve! So $\Omega_{t_\infty} = \Omega_1$, and we have just bubbled off a $(-1)$-curve, as desired!
**Theorem C.** There is a Kähler metric $g$ on $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ which is conformal to an Einstein metric. Moreover, there is a $1$-parameter family $[0, 1) \ni t \mapsto g_t$ of extremal Kähler metrics on $\mathbb{CP}_2 \# 3\overline{\mathbb{CP}}_2$ s.t.

- $g_0$ is Kähler-Einstein, and such that
- $g_{t_j} \to g$ in the Gromov-Hausdorff sense for some $t_j \uparrow 1$. 