

# COURS DE LA CHAIRE D'EXCELLENCE



FONDATION  
SCIENCES  
MATHÉMATIQUES DE  
PARIS



INSTITUT DE MATHÉMATIQUES  
JUSSIEU - PARIS RIVE GAUCHE

## CLAUDE R. LEBRUN

STONY BROOK

ACCUEILLI À L'IMJ-PRG (SORBONNE UNIV., UNIV. PARIS CITÉ, CNRS)

# *EINSTEIN METRICS, FOUR-MANIFOLDS, AND DIFFERENTIAL TOPOLOGY*

AMPHITHÉÂTRE YVONNE CHOQUET-BRUHAT (BÂT. PERRIN)

JEUDI 19 MARS 2026

JEUDI 26 MARS 2026

JEUDI 2 AVRIL 2026

SALLE PIERRE GRISVARD (BÂT. BOREL, 3<sup>E</sup> ÉTAGE)

JEUDI 9 AVRIL 2026

JEUDI 16 AVRIL 2026

JEUDI 23 AVRIL 2026\*

DE 14H À 17H15

INSTITUT HENRI POINCARÉ

11, RUE PIERRE ET MARIE CURIE, 75005 PARIS

\* La durée totale du cours étant de 15h, une séance parmi les trois dernières indiquées sera supprimée, en concertation avec Claude LeBrun.

INFORMATIONS ET RÉSUMÉ DU COURS  
[WWW.SCIENCESMATHS-PARIS.FR](http://WWW.SCIENCESMATHS-PARIS.FR)



*Einstein Metrics,*

*Four-Manifolds, &*

*Differential Topology, V*

Claude LeBrun

Stony Brook University

Cours de la Chaire d'Excellence  
Fondation Sciences Mathématiques de Paris  
Institut Henri Poincaré, jeudi 16 avril 2026

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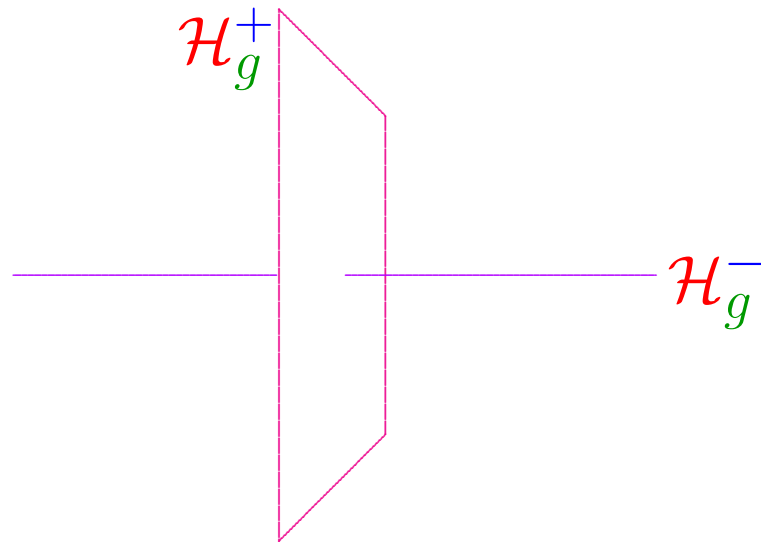
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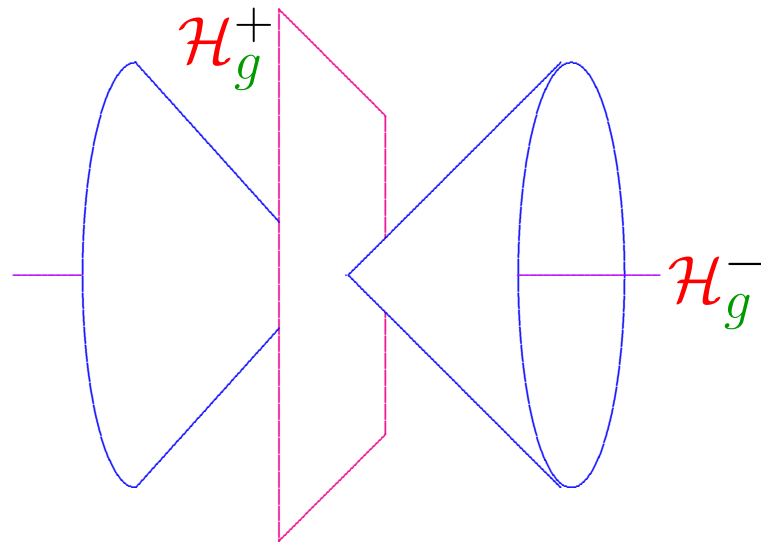
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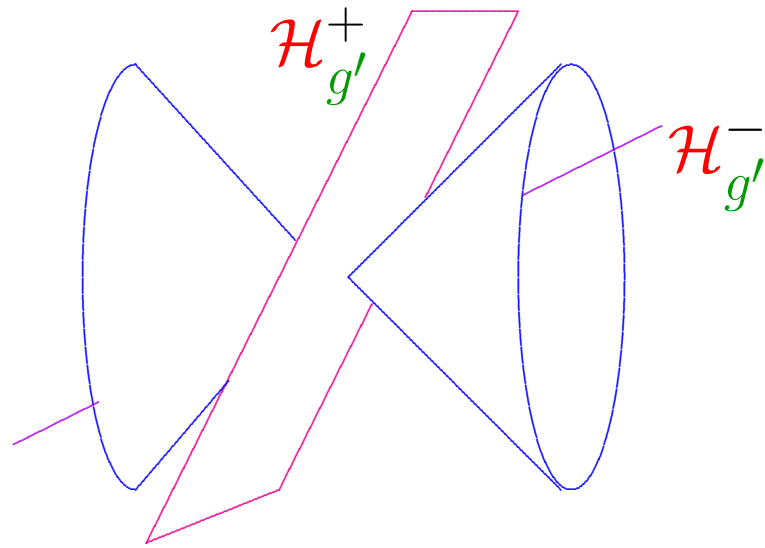
Now use a result of Donaldson on period map.



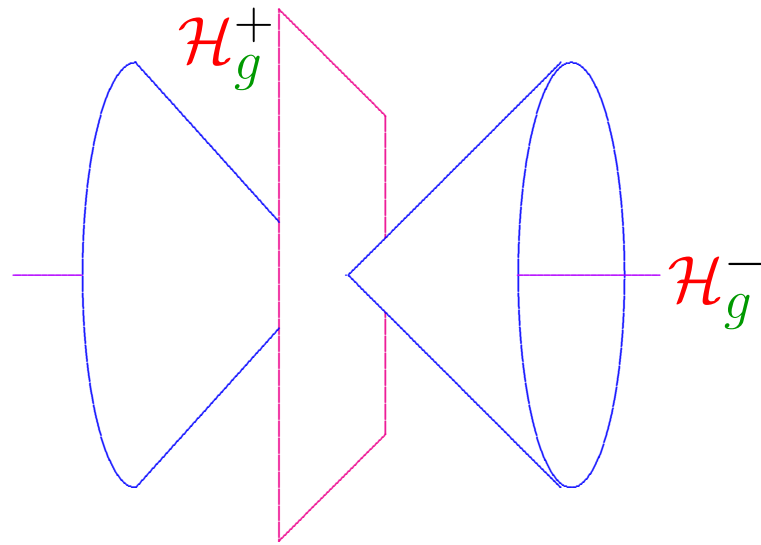
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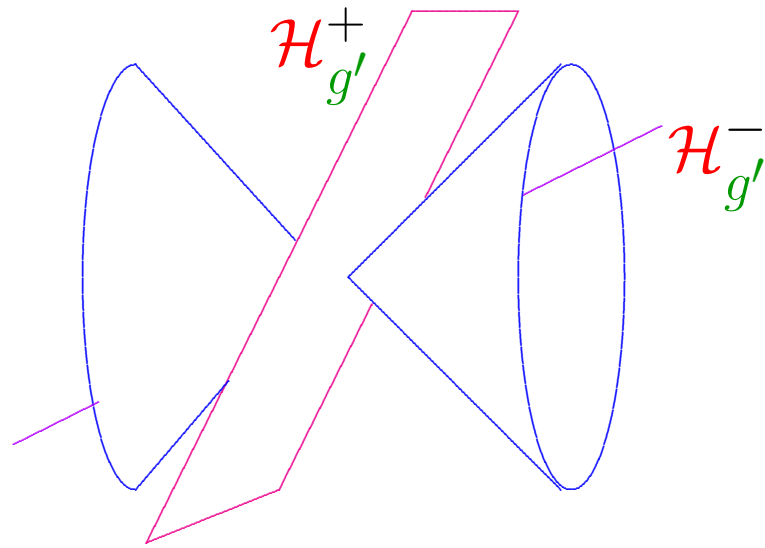
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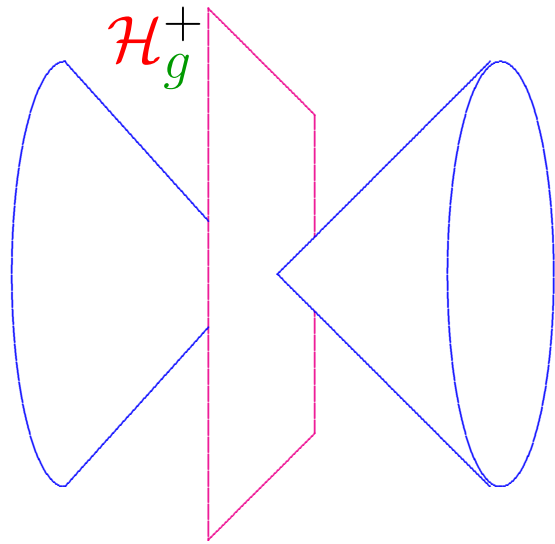
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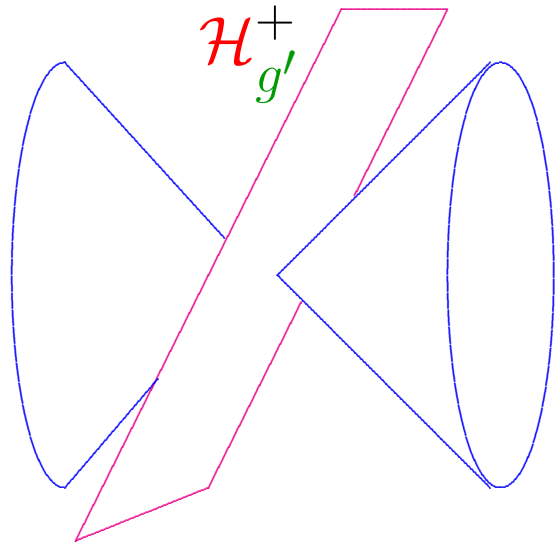
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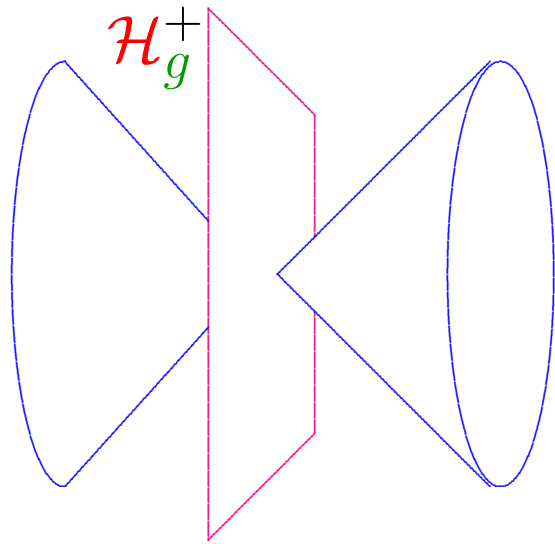
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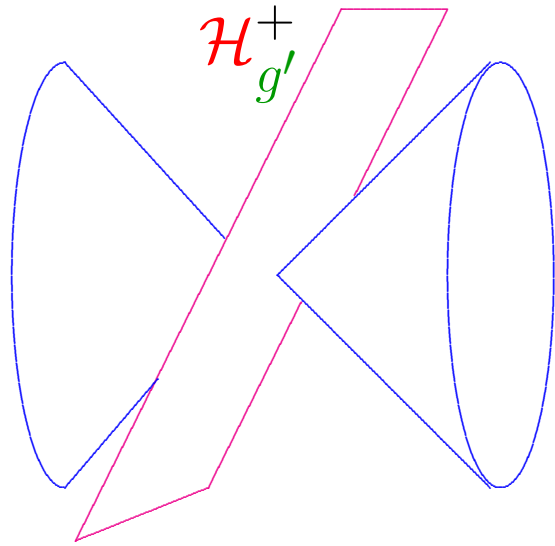
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We may also normalize so that  $v_j^2 = 1$  for each  $j$ .

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Notice that this of course assigns a specific natural number to each differentiable structure.

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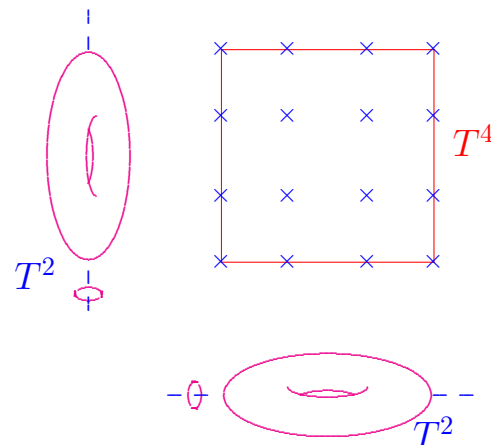
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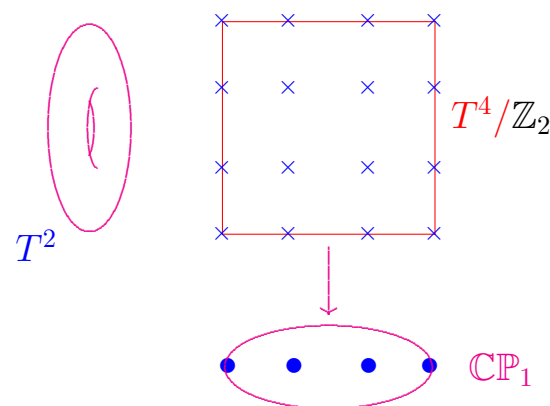
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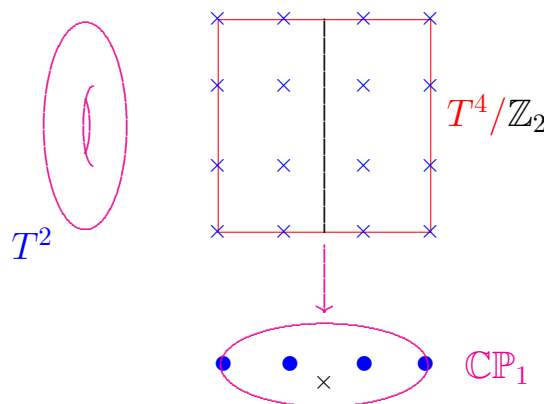
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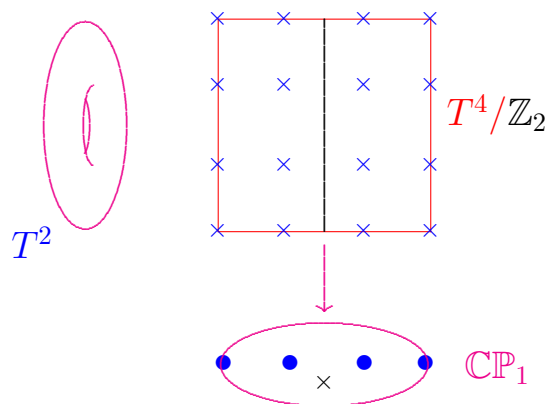
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Replace chosen fiber  $T^2$  with  $T^2/\mathbb{Z}_{2\ell+1}$



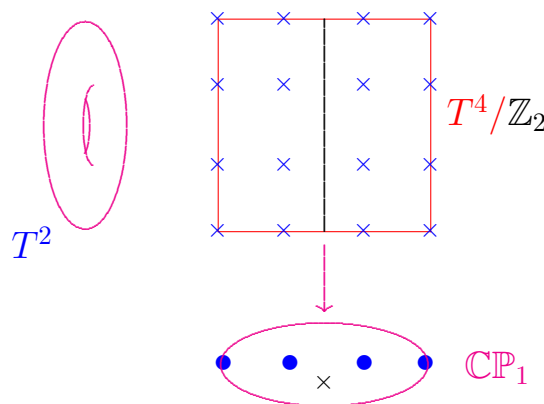
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**Key:**  $[T^2 \times (D^2 - \{0\})] / \mathbb{Z}_{2\ell+1} \cong T^2 \times (D^2 - \{0\})$ .



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As  $\ell \rightarrow \infty$ ,  $BW(M_\ell) \rightarrow \infty$ .

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*with equality only if  $(M, g)$  is finitely covered by flat  $T^4$  or Calabi-Yau  $K3$ .*

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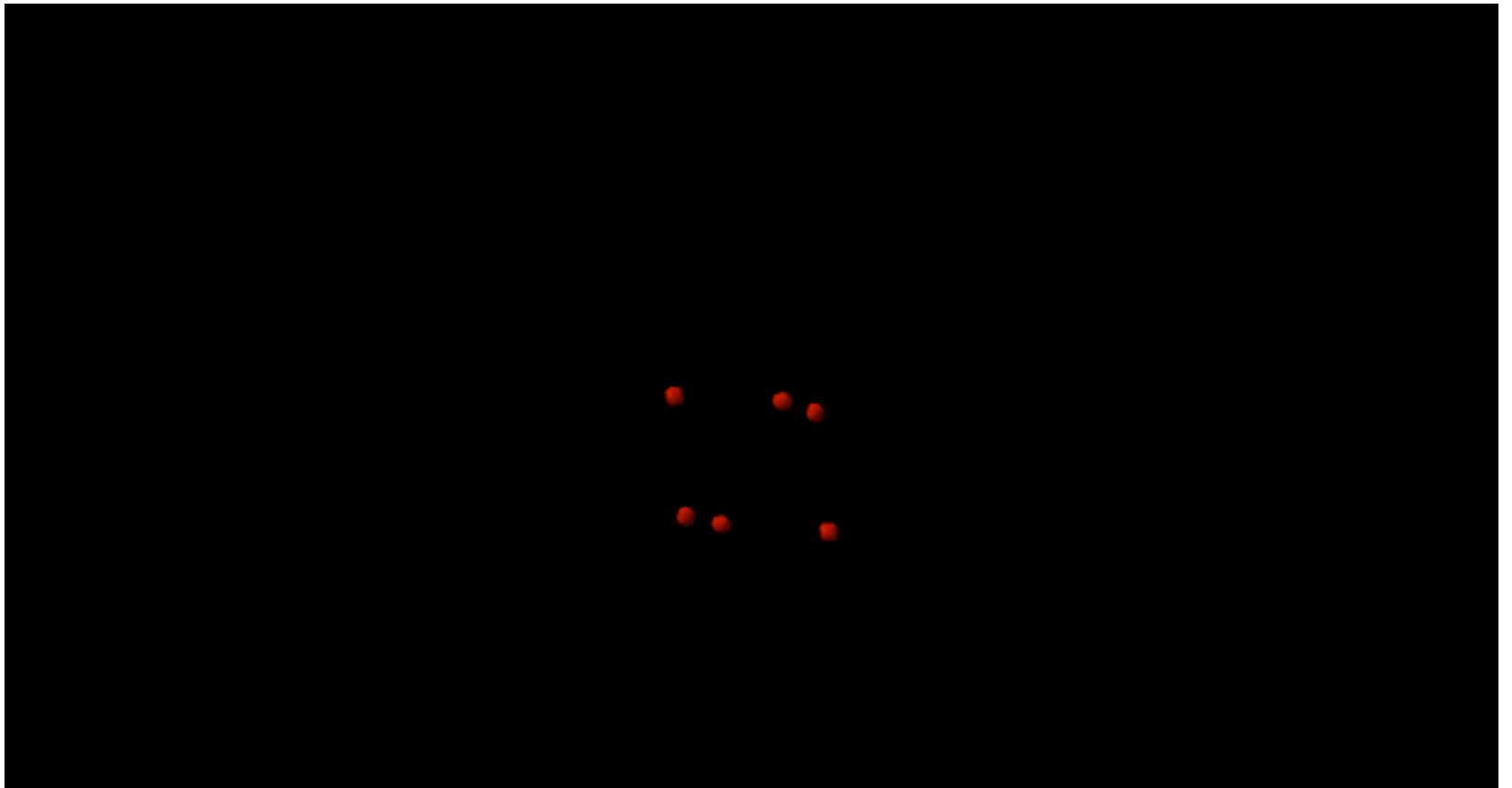
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The finiteness of the set of monopole classes also allows us to make the following definition:

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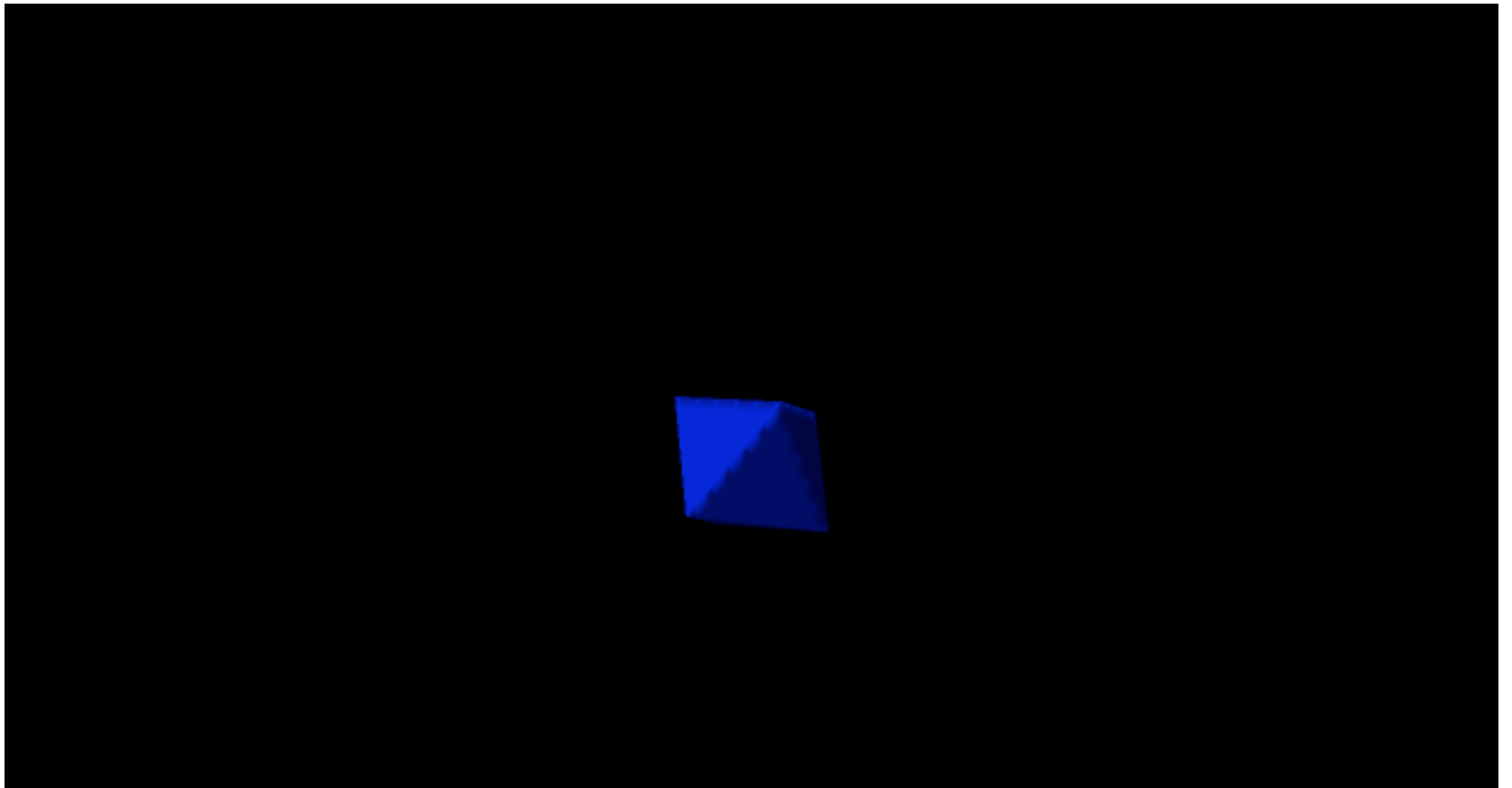


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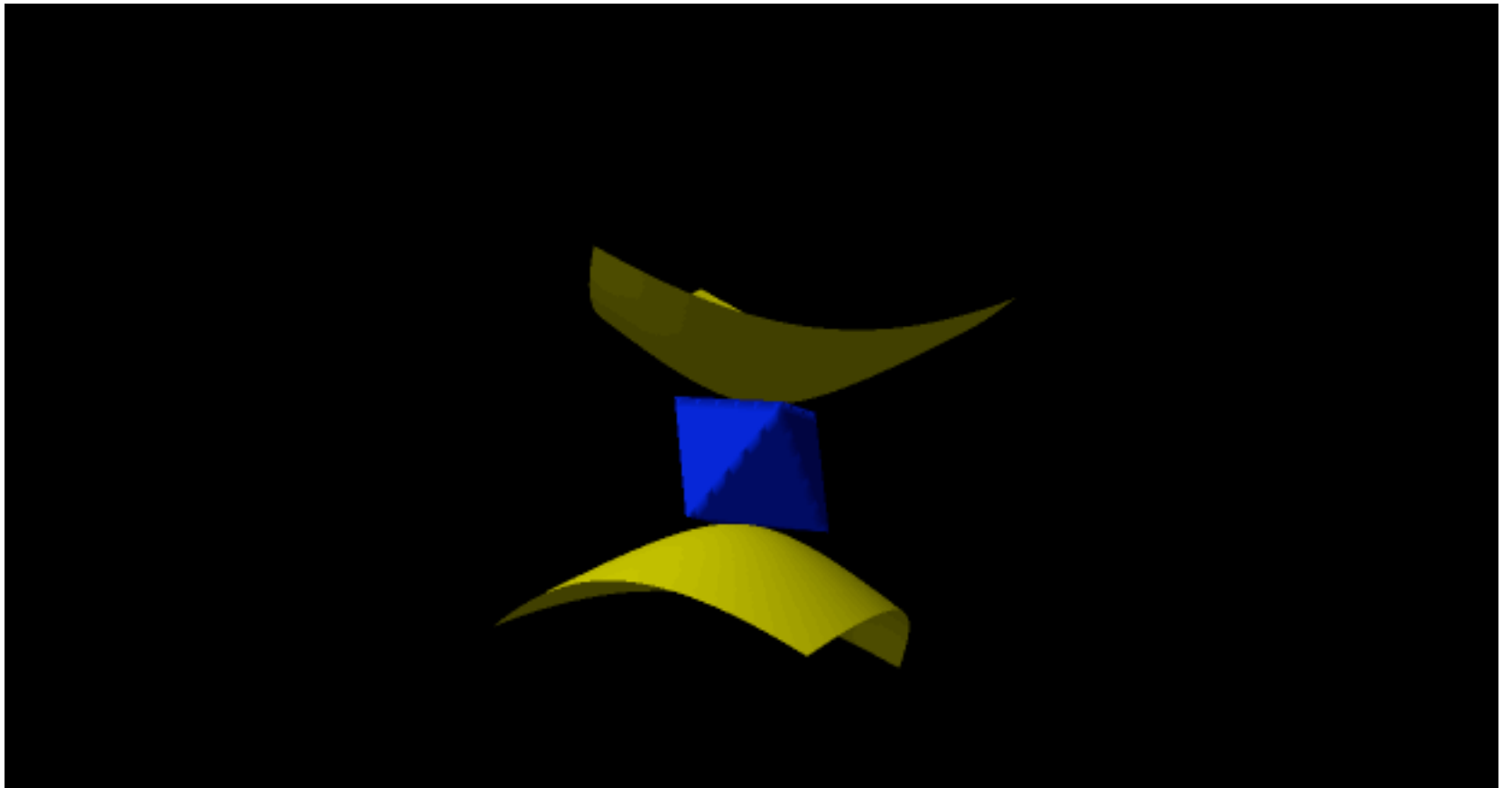
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We may therefore set  $\beta(M) := \sqrt{\beta^2(M)}$ .

**Theorem** (Curvature Estimates). *For any  $C^2$  Riemannian metric  $g$*

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The equality case intimately reflects the following. . .

**Example** If  $X$  is a minimal complex surface with  $b_+ > 1$ , and if

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then

$$\beta^2(M) = c_1^2(X)$$

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Indeed, since every basic class  $\in \mathfrak{E}$ , one can show that  $c_1(X) \in \text{Hull}(\mathfrak{E})$ , and hence

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Bauer-Furuta invariant allows one to show that

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in conjunction with a construction of minimizing sequences of metrics  $g_j$  on this manifold.  $\diamond$

**Theorem.** *Let  $X$ ,  $Y$ , and  $Z$  be compact oriented simply connected symplectic 4-manifolds with  $b_+ \equiv 3 \pmod{4}$ . Then, with respect to the canonical isomorphisms*

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*are monopole classes on  $X \# Y$  and  $X \# Y \# Z$ , respectively. Here the  $\pm$  signs are arbitrary, and independent of one another.*

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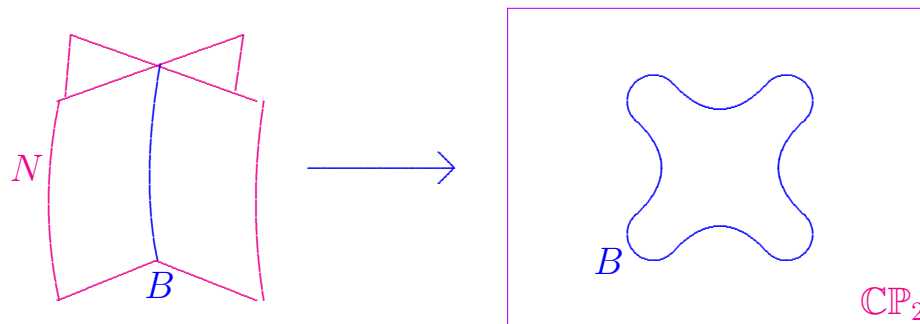
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*Intermission*

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We now turn to the  $\lambda > 0$  case.

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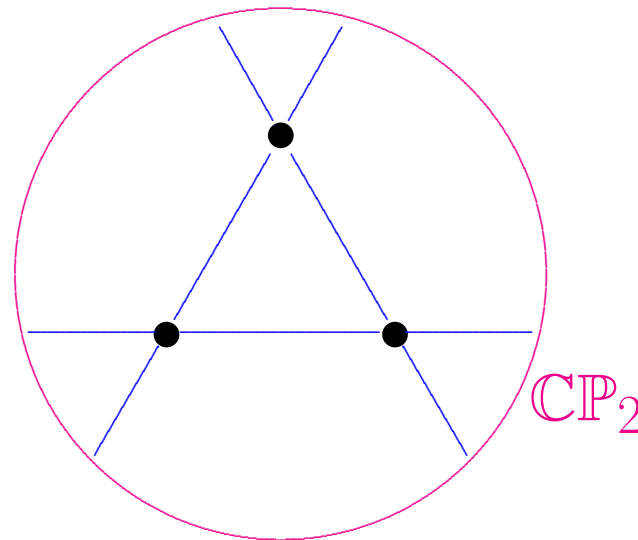
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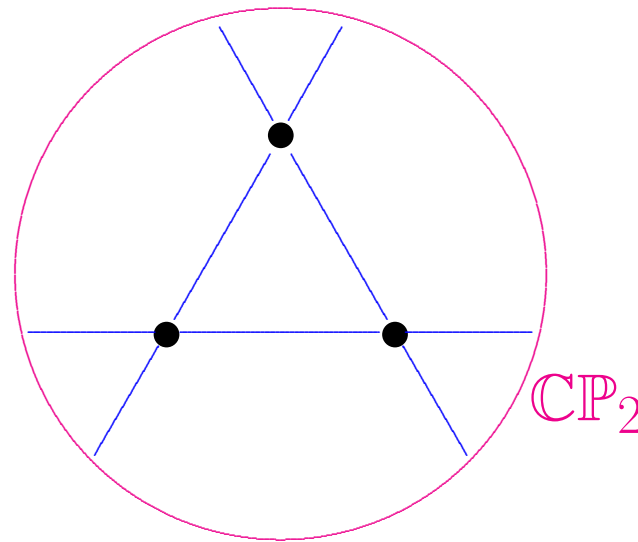
Blow-up of  $\mathbb{C}P_2$  at  $k$  distinct points,  $0 \leq k \leq 8$ ,  
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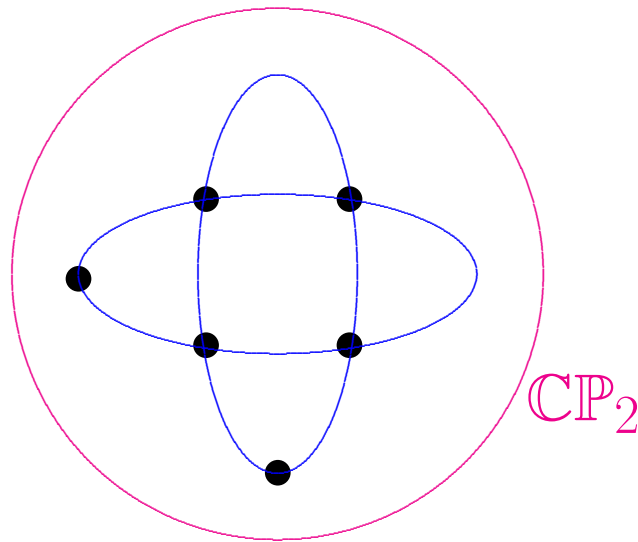


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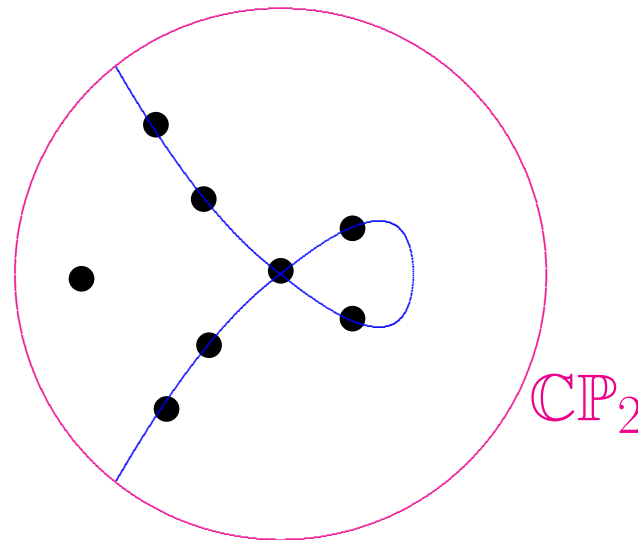


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$$g = fh$$

for some Kähler metric  $h$ , smooth function  $f > 0$ .

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Similarly for  $S^{2n+1} \times S^{2m+1}$ .

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Conformally invariant Riemannian functional:

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Conformally invariant means unchanged by

$$g \rightsquigarrow u^2 g$$

for any smooth function  $u : M \rightarrow \mathbb{R}^+$ .

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$$\nabla^a B_{ab} = 0$$

## 4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W|^2 - \frac{|\dot{r}|^2}{2} \right) d\mu$$

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$\implies$  conformally Einstein metrics are critical, too.

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So what does the identity

$$B_{ab} = 0$$

say about conformally Kähler, Einstein metrics?

Riemann curvature of  $g$

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	$\Lambda^{+*}$	$\Lambda^{-*}$
$\Lambda^+$	$W_+ + \frac{s}{12}$	$\overset{\circ}{r}$
$\Lambda^-$	$\overset{\circ}{r}$	$W_- + \frac{s}{12}$

where

$s$  = scalar curvature

$\overset{\circ}{r}$  = trace-free Ricci curvature

$W_+$  = self-dual Weyl curvature (*conformally invariant*)

$W_-$  = anti-self-dual Weyl curvature //

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The bottom-right cell of the table is shaded with a dotted pattern.

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$$W_+ = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

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X.X. Chen: always minimizers.

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Donaldson/Mabuchi/Chen-Tian:  
unique modulo bihomorphisms.

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and corresponds to **harmonic** primitive  $(1, 1)$ -form

$$\psi := B(J\cdot, \cdot) = \frac{1}{12} \left[ s\rho + 2i\partial\bar{\partial}s \right]_0$$

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So the critical points of restriction of  $\mathcal{W}$  to  $\{\text{Kähler metrics}\}$  must also have  $B = 0$ !

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For global Einstein metric, need to show  $s > 0$ .

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so  $c_1 > 0$ .

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But it then immediately implies...

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**Lemma.** *Suppose that  $h$  is a Bach-flat Kähler metric on a Del Pezzo surface  $(M^4, J)$ . Then the Hermitian metric  $g = s^{-2}h$  is Einstein, with positive Einstein constant.*

**Lemma.** Suppose compact complex surface  $(M^4, J)$  admits a Hermitian  $g$  which is *Einstein*, but not Kähler. Then  $(M^4, J)$  is a *Del Pezzo surface*.

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**Lemma.** Suppose that  $h$  is a Bach-flat Kähler metric on a *Del Pezzo surface*  $(M^4, J)$ . Then the Hermitian metric  $g = s^{-2}h$  is *Einstein*, with positive Einstein constant.

**Lemma.** Conversely, any Hermitian, *Einstein* metric on a *Del Pezzo surface* arises in this way.

**Theorem.** *Let  $(M^4, J)$  be a Del Pezzo surface. Then, up to automorphisms and rescaling, there is a **unique** Bach-flat Kähler metric  $h$  on  $M$ . This metric is characterized by the fact that it minimizes the Calabi functional*

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Hermitian, Einstein metric then given by

$$g = s^{-2}h$$

and uniqueness **Theorem** follows.

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In all other cases, any extremal Kähler metric must have  $s \equiv \text{const}$ , so that any Hermitian, **Einstein** metric must be **Kähler-Einstein**.

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The Kähler-Einstein case is of course quite non-trivial, for complementary reasons. For a watertight treatment, see [Odaka-Spotti-Sun 2016](#).

# Action Function on Kähler Cone

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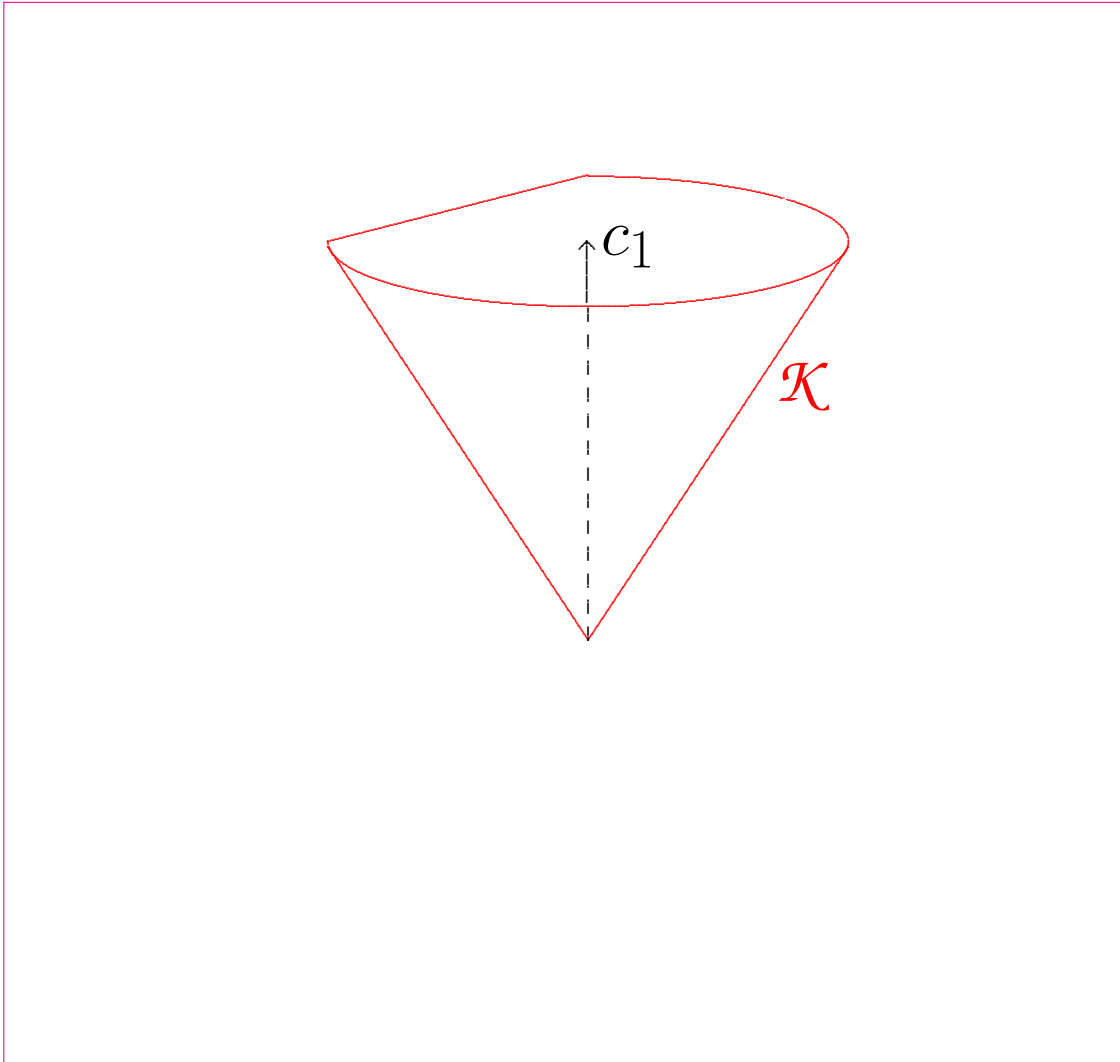
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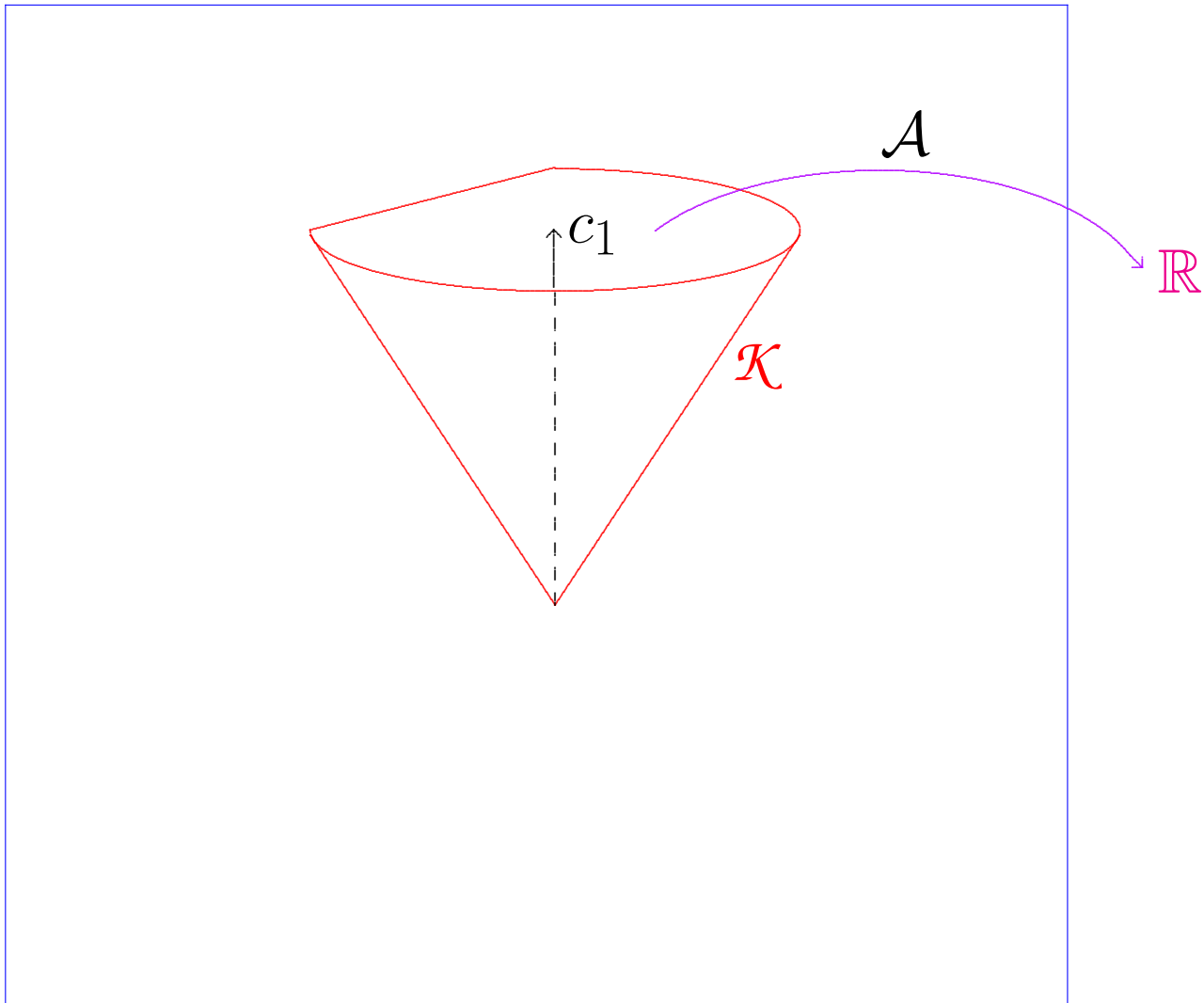
where  $\mathcal{F}$  is Futaki invariant.

$\mathcal{A}$  is function on Kähler cone  $\mathcal{K} \subset H^2(M, \mathbb{R})$ .



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$$

( $M$  Del Pezzo)



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## Action Function on Kähler Cone

For any extremal Kähler  $(M^4, h, J)$ ,

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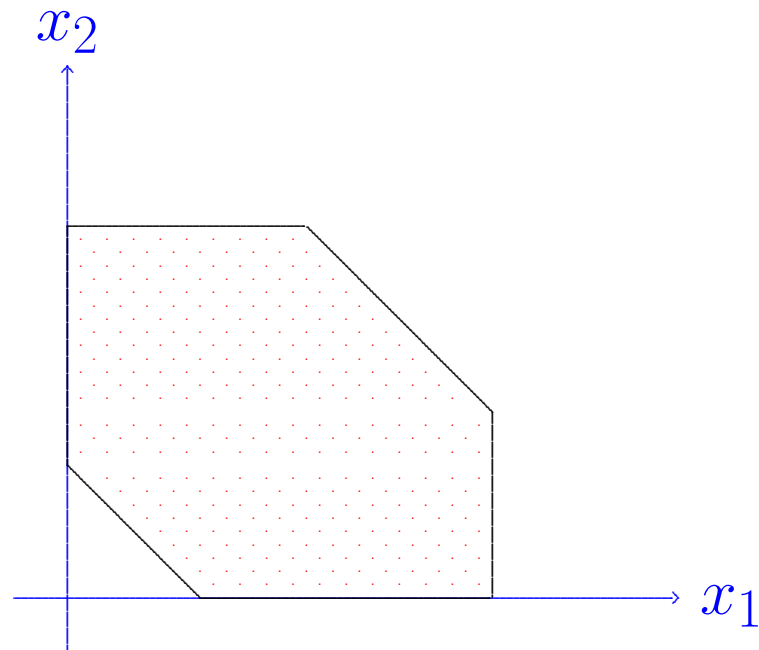
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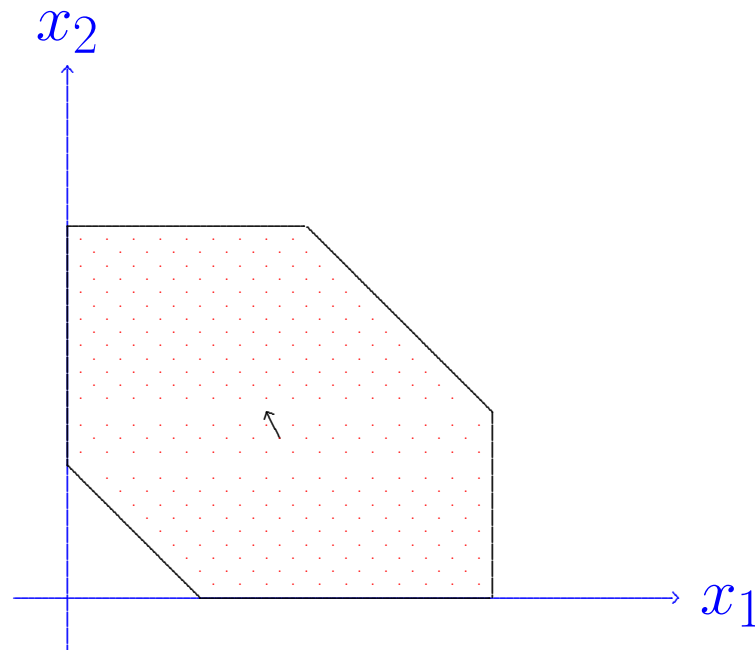
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$$\begin{aligned}
& 3[3 + 28\gamma + 96\gamma^2 + 168\gamma^3 + 164\gamma^4 + 80\gamma^5 + 16\gamma^6 + 16\beta^6(1 + \gamma)^4 + 16\alpha^6(1 + \beta + \gamma)^4 + 16\beta^5(5 + 24\gamma + 43\gamma^2 + 37\gamma^3 + 15\gamma^4 + 2\gamma^5) + 4\beta^4(41 + 228\gamma + 478\gamma^2 + 496\gamma^3 + 263\gamma^4 + \\
& 60\gamma^5 + 4\gamma^6) + 8\beta^3(21 + 135\gamma + 326\gamma^2 + 392\gamma^3 + 248\gamma^4 + 74\gamma^5 + 8\gamma^6) + 4\beta(7 + 58\gamma + 176\gamma^2 + 270\gamma^3 + 228\gamma^4 + 96\gamma^5 + 16\gamma^6) + 4\beta^2(24 + 176\gamma + 479\gamma^2 + 652\gamma^3 + 478\gamma^4 + \\
& 172\gamma^5 + 24\gamma^6) + 16\alpha^5(5 + 2\beta^5 + 24\gamma + 43\gamma^2 + 37\gamma^3 + 15\gamma^4 + 2\gamma^5 + \beta^4(15 + 14\gamma) + \beta^3(37 + 70\gamma + 30\gamma^2) + \beta^2(43 + 123\gamma + 108\gamma^2 + 30\gamma^3) + \beta(24 + 92\gamma + 123\gamma^2 + 70\gamma^3 + \\
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& 556\gamma^3 + 98\gamma^4) + 4\beta(57 + 263\gamma + 438\gamma^2 + 338\gamma^3 + 119\gamma^4 + 14\gamma^5)) + 8\alpha^3(21 + 135\gamma + 326\gamma^2 + 392\gamma^3 + 248\gamma^4 + 74\gamma^5 + 8\gamma^6 + 8\beta^6(1 + \gamma) + 2\beta^5(37 + 70\gamma + 30\gamma^2) + 4\beta^4(62 + \\
& 169\gamma + 139\gamma^2 + 35\gamma^3) + 4\beta^3(98 + 353\gamma + 428\gamma^2 + 210\gamma^3 + 35\gamma^4) + 2\beta^2(163 + 735\gamma + 1179\gamma^2 + 856\gamma^3 + 278\gamma^4 + 30\gamma^5) + \beta(135 + 736\gamma + 1470\gamma^2 + 1412\gamma^3 + 676\gamma^4 + 140\gamma^5 + \\
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& 2\beta^3(135 + 736\gamma + 1470\gamma^2 + 1412\gamma^3 + 676\gamma^4 + 140\gamma^5 + 8\gamma^6) + 4\beta^2(44 + 278\gamma + 645\gamma^2 + 735\gamma^3 + 438\gamma^4 + 123\gamma^5 + 12\gamma^6) + 2\beta(29 + 210\gamma + 556\gamma^2 + 736\gamma^3 + 526\gamma^4 + 184\gamma^5 + \\
& 24\gamma^6)) + 4\alpha^2(24 + 176\gamma + 479\gamma^2 + 652\gamma^3 + 478\gamma^4 + 172\gamma^5 + 24\gamma^6 + 24\beta^6(1 + \gamma)^2 + 4\beta^5(43 + 123\gamma + 108\gamma^2 + 30\gamma^3) + 2\beta^4(239 + 876\gamma + 1089\gamma^2 + 556\gamma^3 + 98\gamma^4) + 4\beta^3(163 + \\
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& [1 + 10\gamma + 36\gamma^2 + 64\gamma^3 + 60\gamma^4 + 24\gamma^5 + 24\beta^5(1 + \gamma)^5 + 24\alpha^5(1 + \beta + \gamma)^5 + 12\beta^4(1 + \gamma)^2(5 + 20\gamma + 23\gamma^2 + 10\gamma^3) + 16\beta^3(4 + 28\gamma + 72\gamma^2 + 90\gamma^3 + 57\gamma^4 + 15\gamma^5) + \\
& 12\beta^2(3 + 24\gamma + 69\gamma^2 + 96\gamma^3 + 68\gamma^4 + 20\gamma^5) + 2\beta(5 + 45\gamma + 144\gamma^2 + 224\gamma^3 + 180\gamma^4 + 60\gamma^5) + 12\alpha^4(1 + \beta + \gamma)^2(5 + 20\gamma + 23\gamma^2 + 10\gamma^3 + 10\beta^3(1 + \gamma) + \beta^2(23 + 46\gamma + \\
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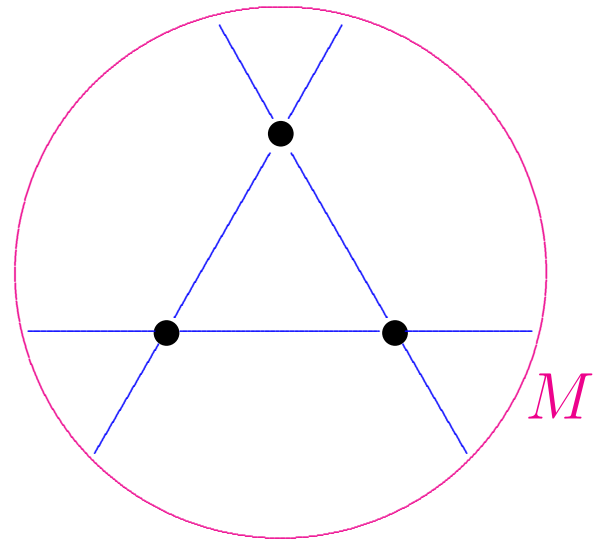
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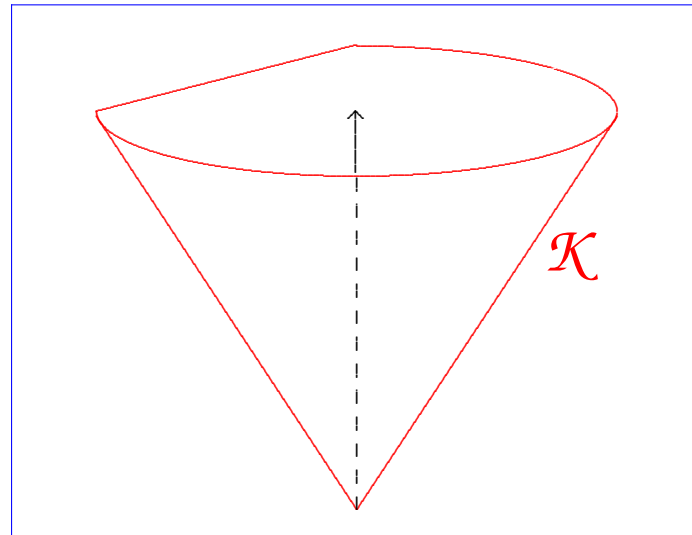
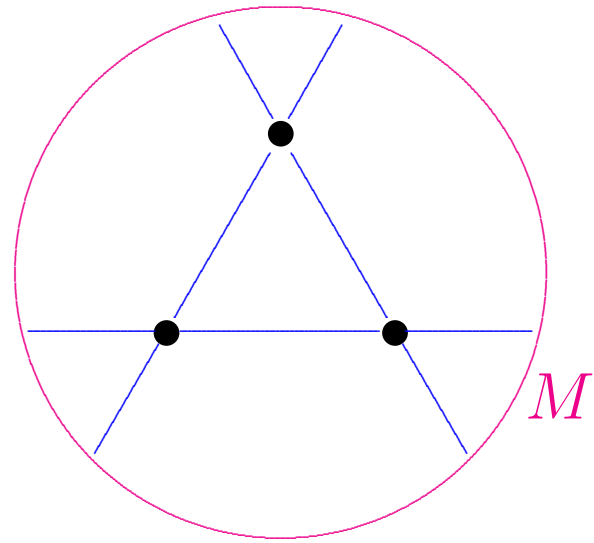
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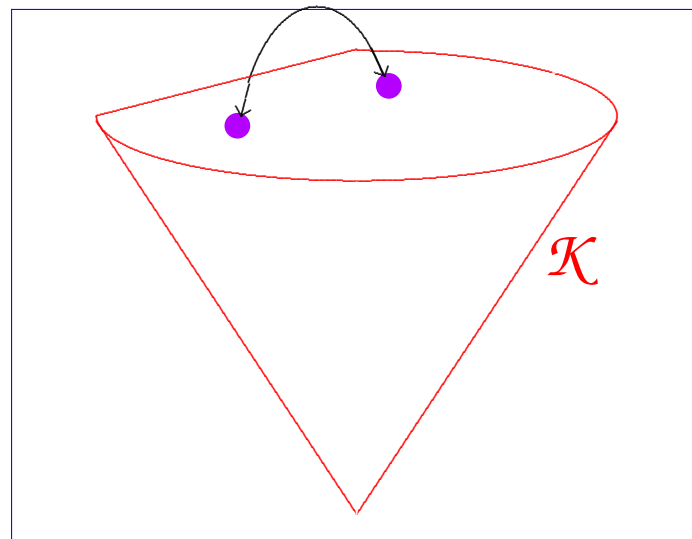
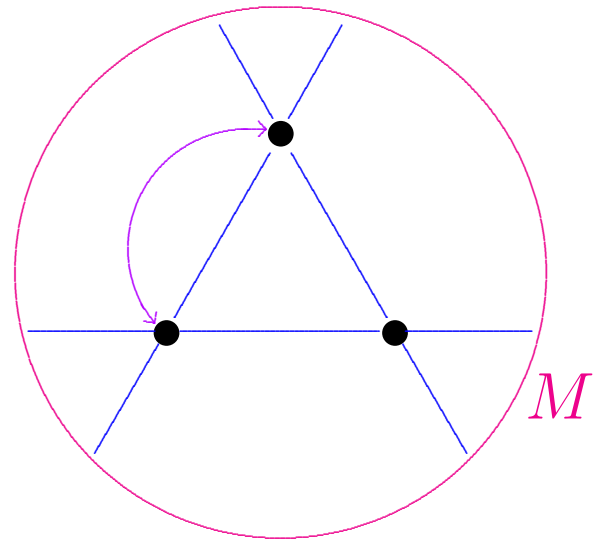
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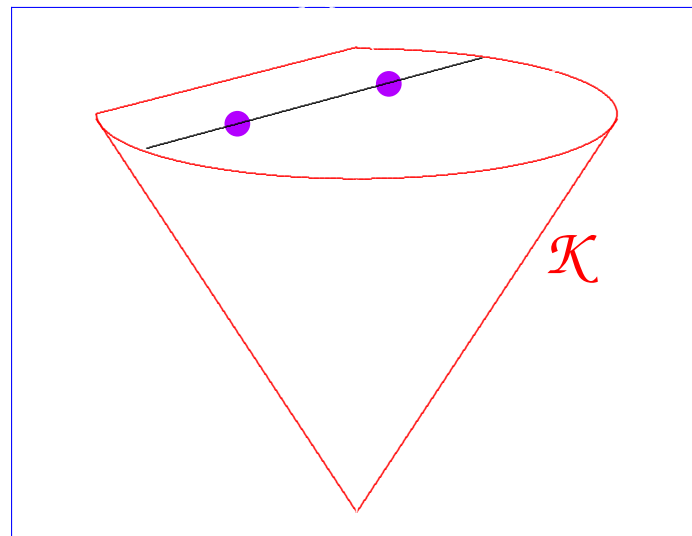
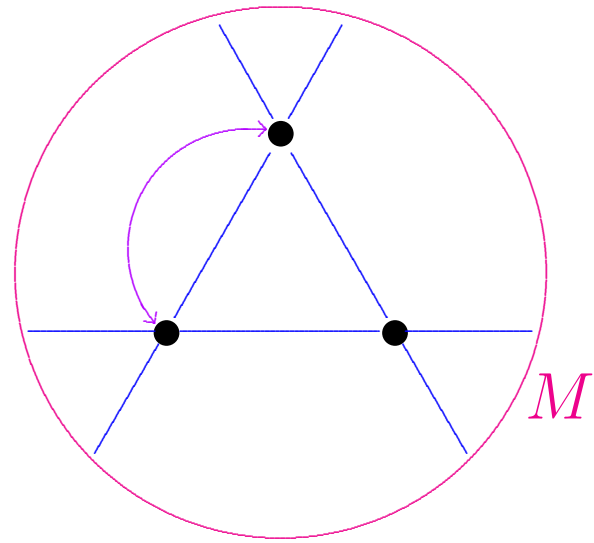
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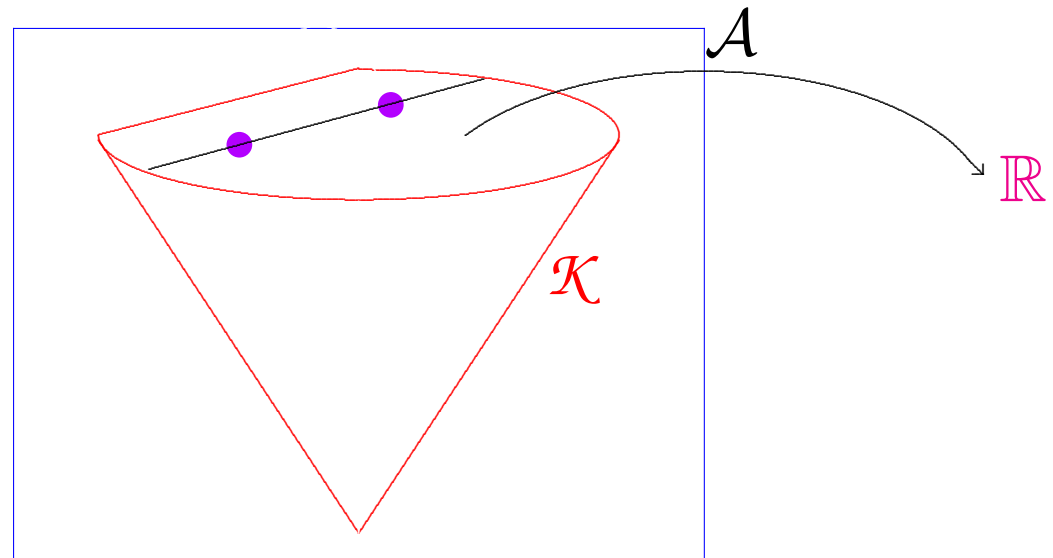
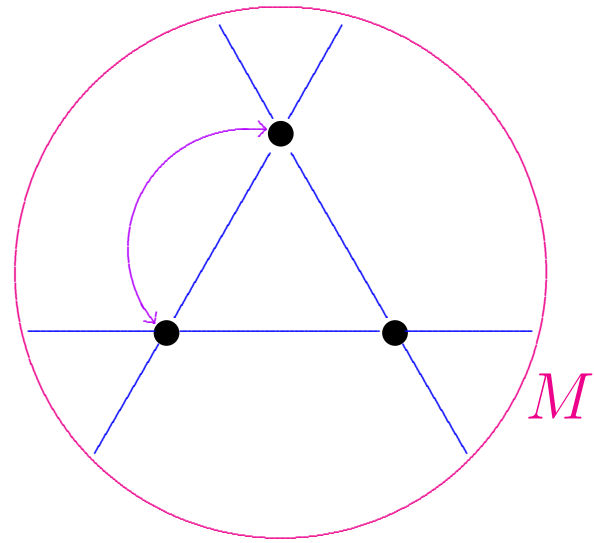
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Final step then just calculus in one variable...

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But by deforming this metric, one can show that every Kähler class in a specific low-energy conical region on  $\mathbb{C}P_2 \# 3\overline{\mathbb{C}P}_2$  is represented by an **extremal Kähler metric**.

**Theorem.** *Let  $M = \mathbb{C}P_2 \# 3\overline{\mathbb{C}P_2}$  be the blow-up of  $\mathbb{C}P_2$  at three non-collinear points, and let*

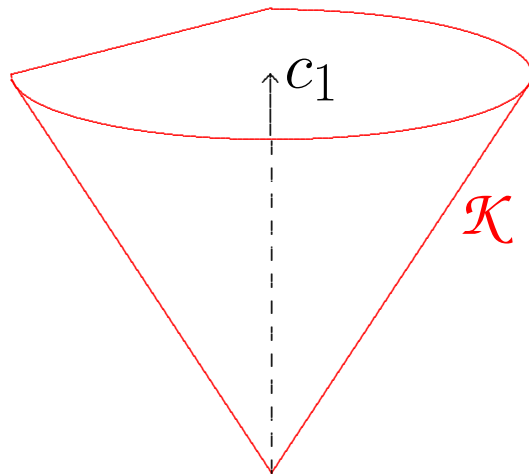
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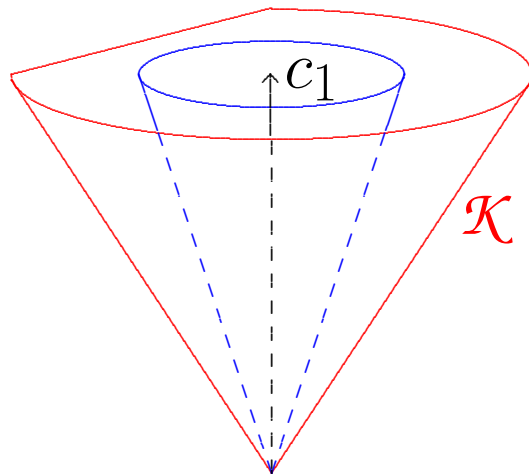
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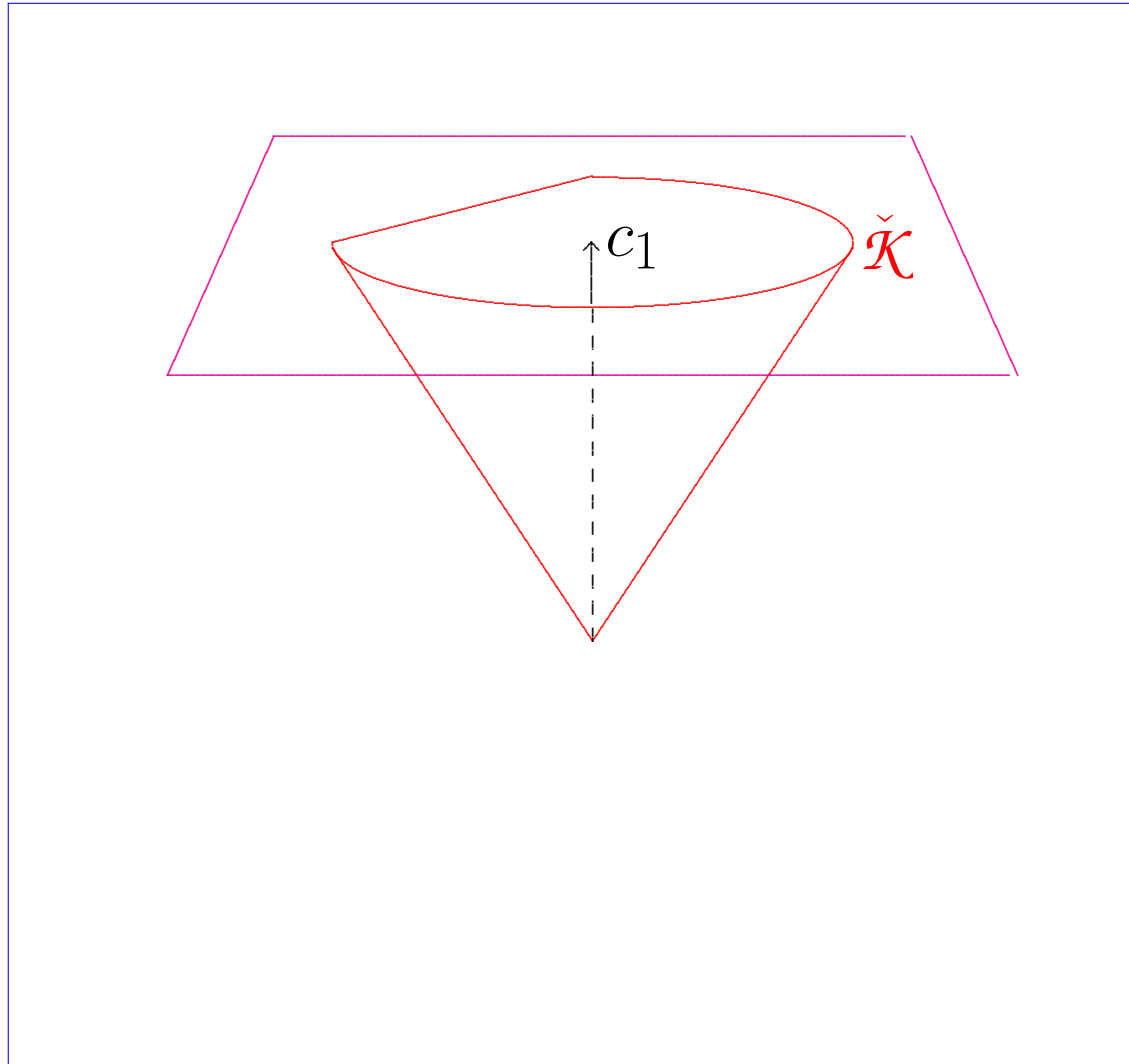
Then there is an extremal Kähler metric  $h$  on  $M$  with Kähler form  $\omega \in [\omega]$ .



$$H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$$



$$\mathcal{T}([\omega]) = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \leq \text{const}$$



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The following result is crucial for this purpose:

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Smallest constant such that

$$\|u\|_{L^4}^2 \leq C_S \left( \|\nabla u\|_{L^2}^2 + V^{-1/2} \|u\|_{L^2}^2 \right)$$

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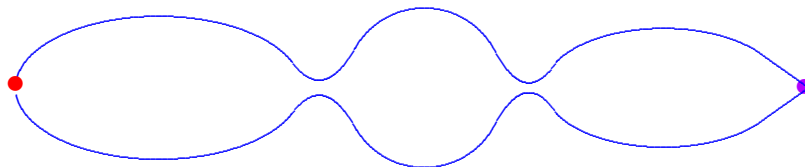
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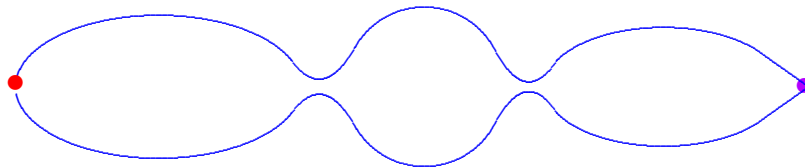
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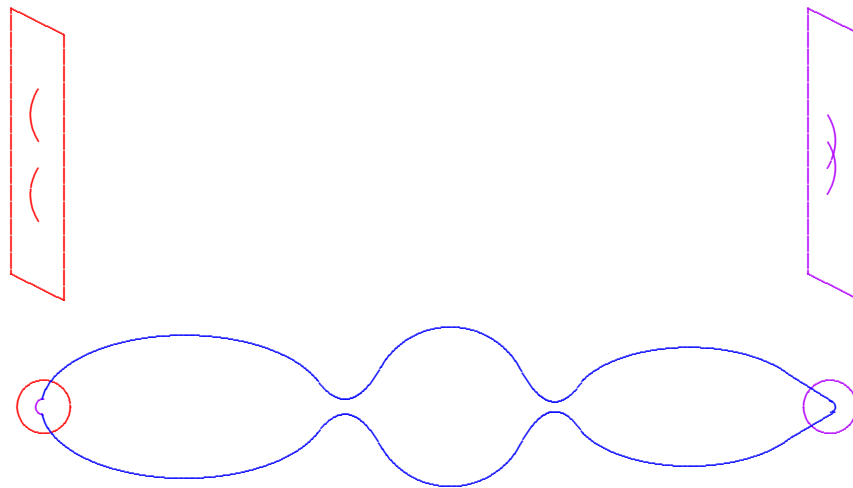
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Suggests continuity method...

**Theorem.** Let  $h_i$  be an arbitrary sequence of unit-volume extremal Kähler metrics on Del Pezzo  $M$  which satisfy

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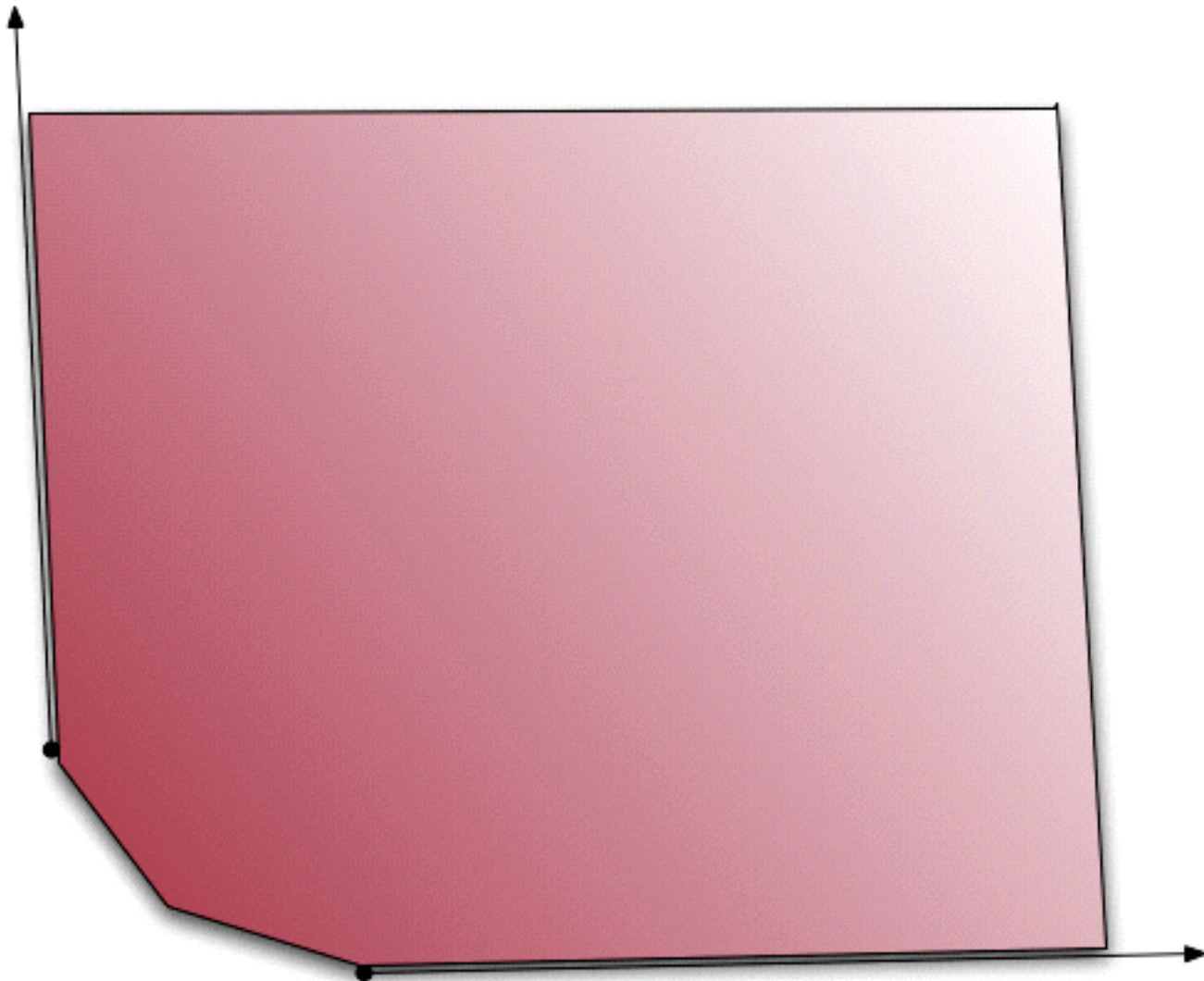
Then  $\exists$  subsequence which Gromov-Hausdorff converges to an extremal Kähler metric on a compact complex 2-orbifold.

Difficulty: rule out deepest bubbles.

**Lemma.** *If  $M$  is toric, any deepest bubble  $(X, h_\infty)$  must be toric, too, with  $H_2(X, \mathbb{R}) \neq 0$  generated by holomorphic  $\mathbb{C}P_1$ 's.*

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Moment map profile:



- Bubble arises by **rescaling** region of manifold by scales  $\nearrow \infty$ .

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Now suppose that  $c_1 \in \mathcal{K}$  on toric Del Pezzo  $M$  represented by extremal Kähler metric. Suppose that  $[\omega]$  satisfies

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$$\begin{aligned} (u_j c_1 + v_j \Omega) \cdot [S] &> 0, \quad \exists u_j, v_j > 0 \\ [S] \cdot [S] &= -k < 0 \\ c_1 \cdot [S] &= 2 - k \\ \Omega \cdot [S] &= 0 \end{aligned}$$

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It follows that bubbling off cannot occur!

One consequence: only Hermitian, Einstein metric on  $\mathbb{C}P_2 \# \overline{3\mathbb{C}P_2}$  is the Kähler-Einstein metric first discovered by Siu!

But by deforming this metric, one can show that every Kähler class in a specific low-energy conical region on  $\mathbb{C}P_2 \# \overline{3\mathbb{C}P_2}$  is represented by an **extremal Kähler metric**.

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**Theorem.** Let  $M = \mathbb{C}P_2 \# 3\overline{\mathbb{C}P_2}$  be the blow-up of  $\mathbb{C}P_2$  at three non-collinear points, and let  $[\omega]$  be a Kähler class on  $M$  for which

$$\mathcal{T}([\omega]) := \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \leq \frac{3}{2}c_1^2 - \frac{1}{4} = c_1^2 + 2.75.$$

Then there is an extremal Kähler metric  $h$  on  $M$  with Kähler form  $\omega \in [\omega]$ .

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Also works when approaching boundary of Kähler cone, but can bubble off  $(-1)$ -curves.

**Theorem.** *Let  $\Omega$  be any Kähler class on  $\mathbb{C}P_2 \# 2\overline{\mathbb{C}P_2}$  for which*

$$\mathcal{T}(\Omega) < 8.75 = c_1^2 + 1.75 .$$

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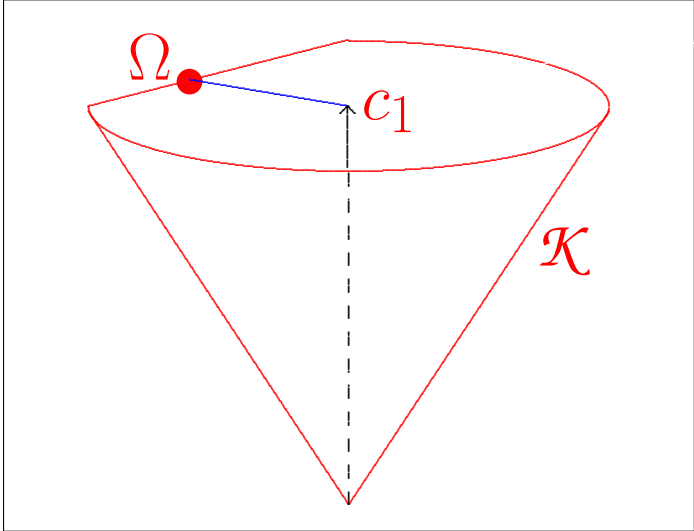
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- $h_0$  is Kähler-Einstein, and such that
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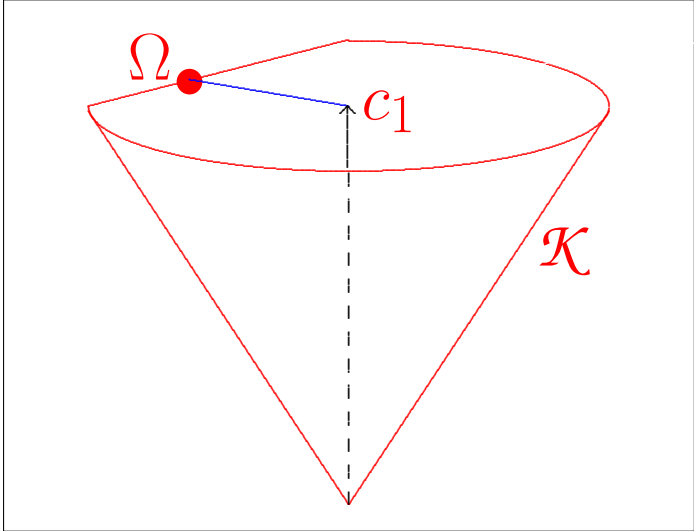
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Similarly for  $\mathbb{C}P_2 \# \overline{\mathbb{C}P_2}$ , though less interesting...

One consequence: only Hermitian, Einstein metric on  $\mathbb{C}P_2 \# 3\overline{\mathbb{C}P}_2$  is the Kähler-Einstein metric first discovered by Siu!

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This approach was carried out in L '13, and gives a technically simpler alternative to the previous existence proof given in Chen-L-Weber '08.

**Proposition.** *Modulo rescalings and biholomorphisms, there is exactly one conformally Kähler, Einstein metric  $g$  on  $M = \mathbb{C}P_2 \# \overline{\mathbb{C}P_2}$ . This metric coincides with the Page metric.*

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**Proposition.** *Modulo rescalings and biholomorphisms, there is only one conformally Kähler, Einstein metric  $g$  on  $M = \mathbb{C}P_2 \# 3\overline{\mathbb{C}P_2}$ . This metric is actually *Kähler-Einstein*, and is exactly the metric discovered by Siu.*

In particular, our existence results prove...

**Theorem.** Suppose that  $M$  is a smooth compact oriented 4-manifold which admits a complex structure  $J$ . Then  $M$  also admits an (unrelated) Einstein metric  $g$  with  $\lambda > 0$

$$\iff M \approx \begin{cases} \mathbb{C}P_2 \#^k \overline{\mathbb{C}P_2}, & 0 \leq k \leq 8, \\ \text{or} \\ S^2 \times S^2 \end{cases}$$

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But combining this with results in symplectic geometry, we also obtain a symplectic analog...

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Similar ideas also yield analogous results concerning Einstein metrics with  $\lambda \geq 0$ :

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Del Pezzo surfaces,

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**Theorem.** Suppose that  $M$  is a smooth compact oriented 4-manifold which admits a symplectic form  $\omega$ . Then  $M$  also admits an Einstein metric  $g$  with  $\lambda \geq 0$  if and only if

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By contrast, the  $\lambda < 0$  case is only partly settled:

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In particular,  $M$  carries Kähler metrics, and therefore admits symplectic structures.

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General type: Minimal model  $X$  satisfies

$$c_1^2(X) > 0$$

$$c_1(X) \cdot [\omega_X] < 0.$$

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If so, quite different from Kähler-Einstein metrics!

# For further reading:

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