Cours de la chaire d'excellence

FONDATION SCIENCES MATHÉMATIQUES DE PARIS

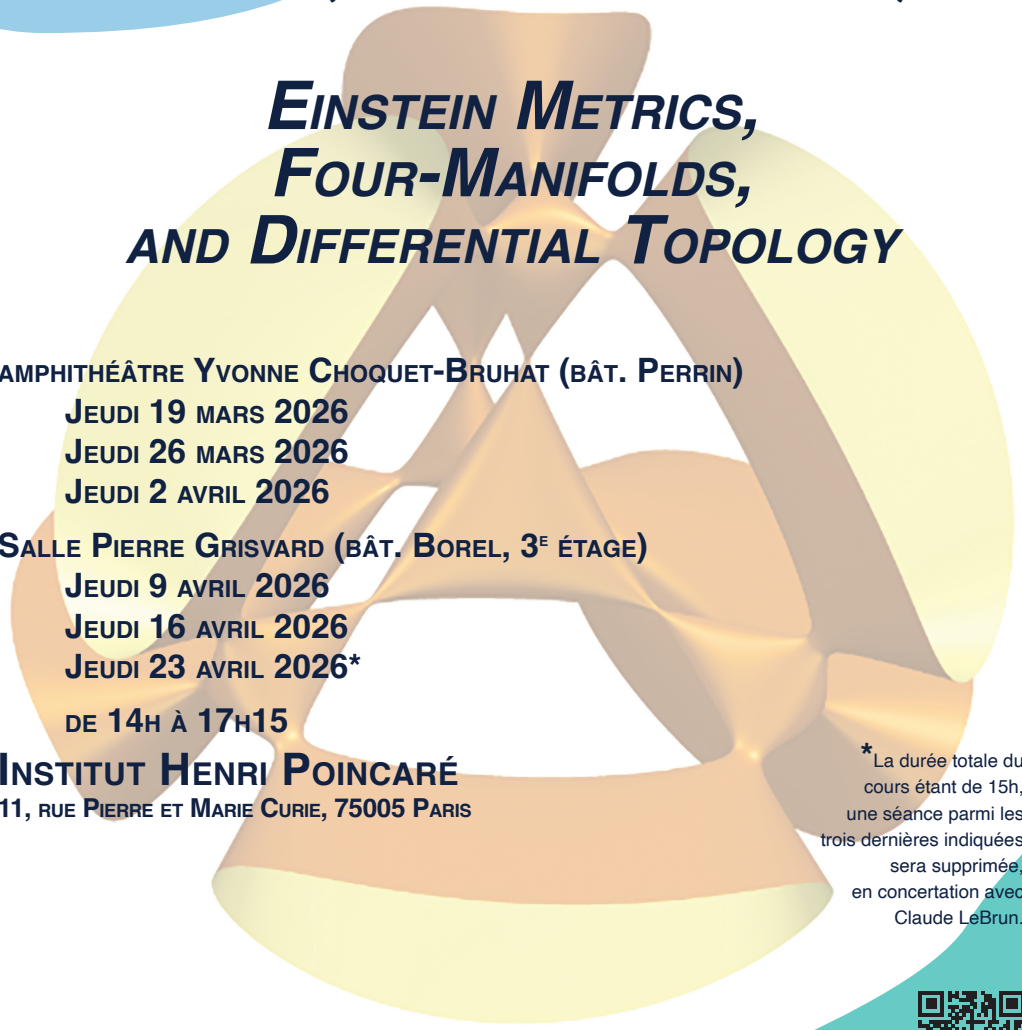
INSTITUT DE MATHÉMATIQUES JUSSIEU - PARIS RIVE GAUCHE

Claude R. LeBrun

Stony Brook

Accueilli à l'IMJ-PRG (Sorbonne Univ., Univ. Paris Cité, CNRS)

Einstein Metrics, Four-Manifolds, and Differential Topology



Amphithéâtre Yvonne Choquet-Bruhat (bât. Perrin)

Jeudi 19 mars 2026

Jeudi 26 mars 2026

Jeudi 2 avril 2026

Salle Pierre Grisvard (bât. Borel, 3e étage)

Jeudi 9 avril 2026

Jeudi 16 avril 2026

Jeudi 23 avril 2026*

de 14h à 17h15

Institut Henri Poincaré

11, rue Pierre et Marie Curie, 75005 Paris

*La durée totale du cours étant de 15h, une séance parmi les trois dernières indiquées sera supprimée, en concertation avec Claude LeBrun.

Informations et résumé du cours

www.sciencesmaths-paris.fr

Einstein Metrics, Four-Manifolds, & Differential Topology, IV

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Dirac Operators and Scalar Curvature

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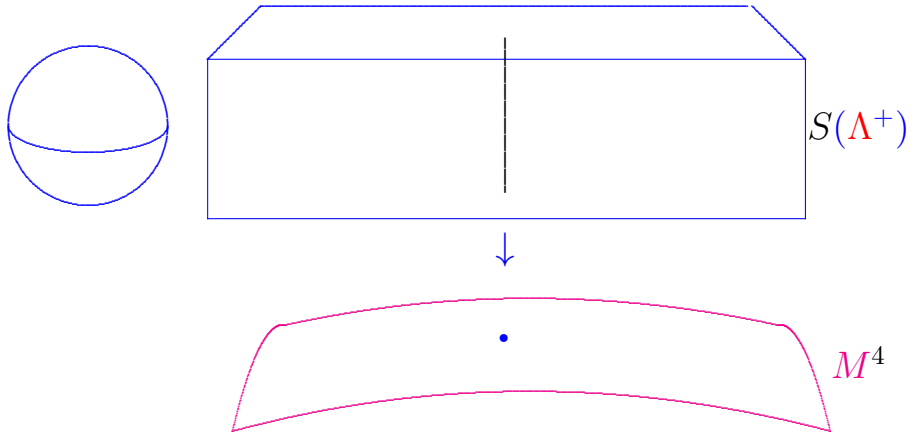

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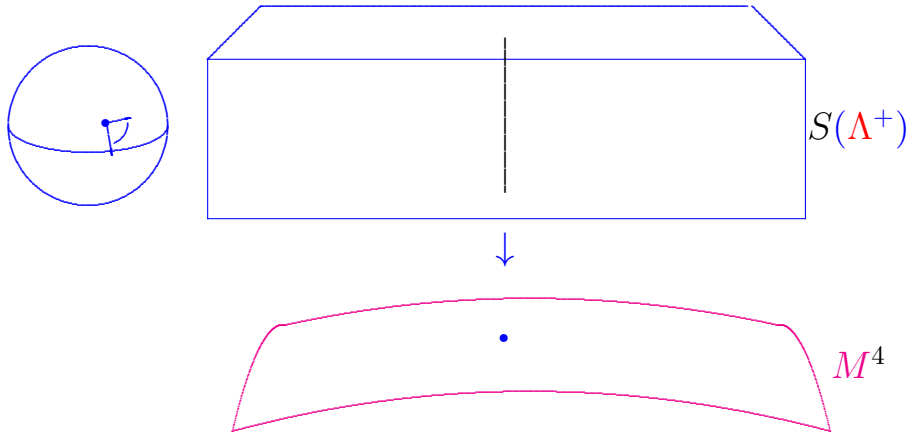

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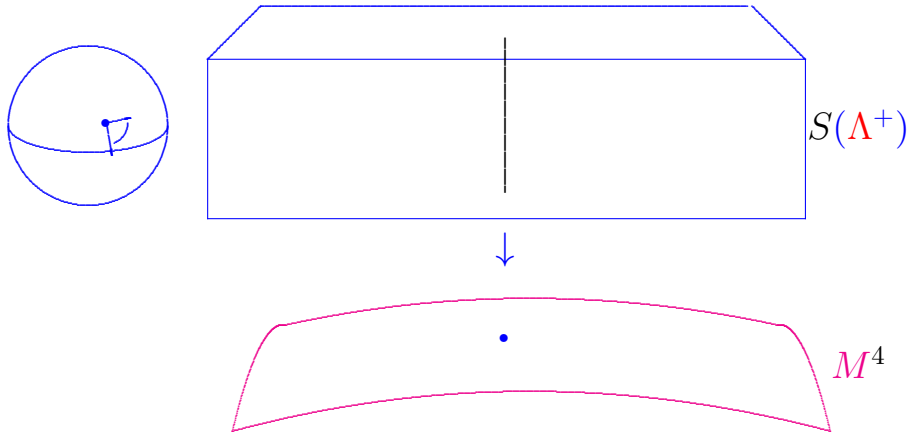

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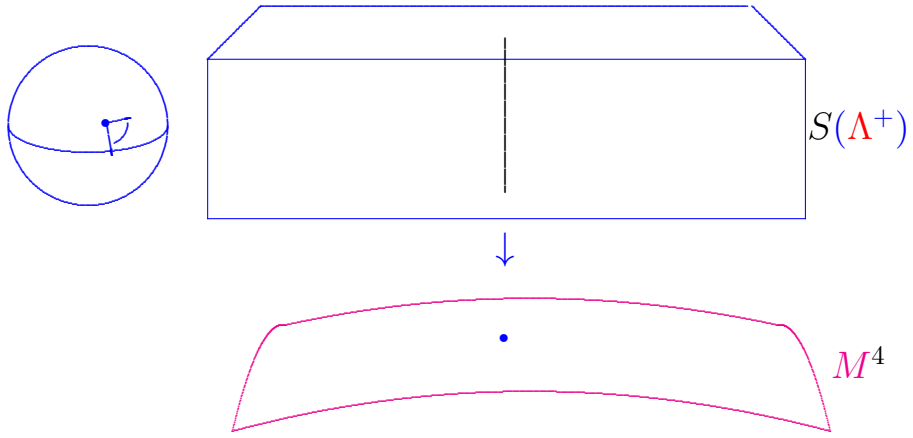

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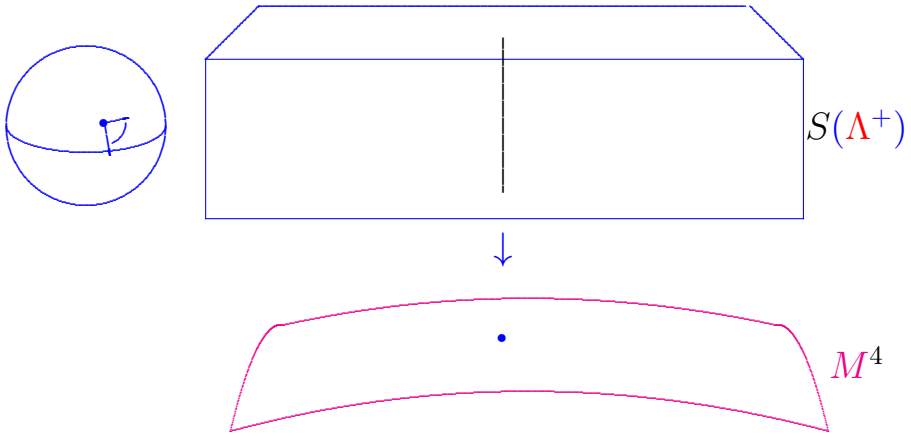

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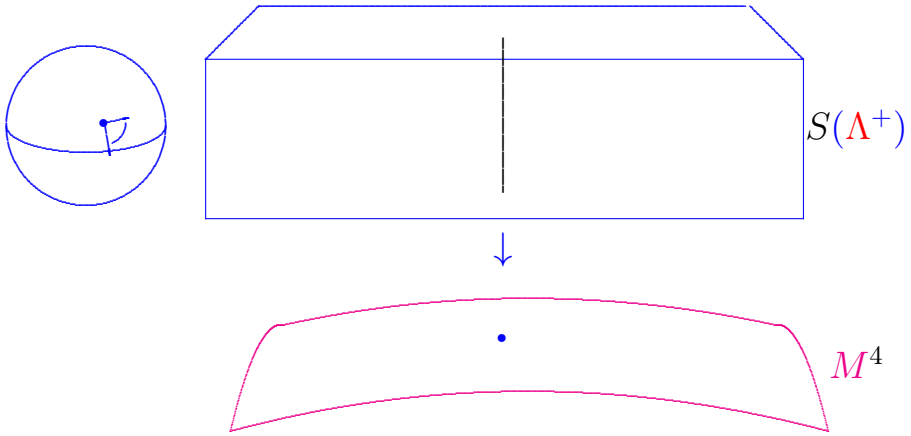

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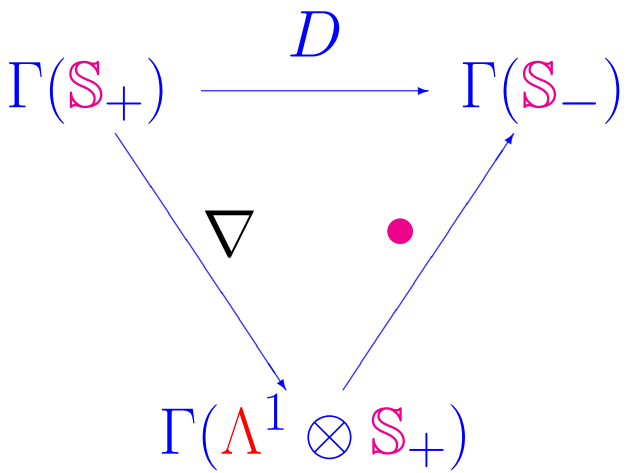

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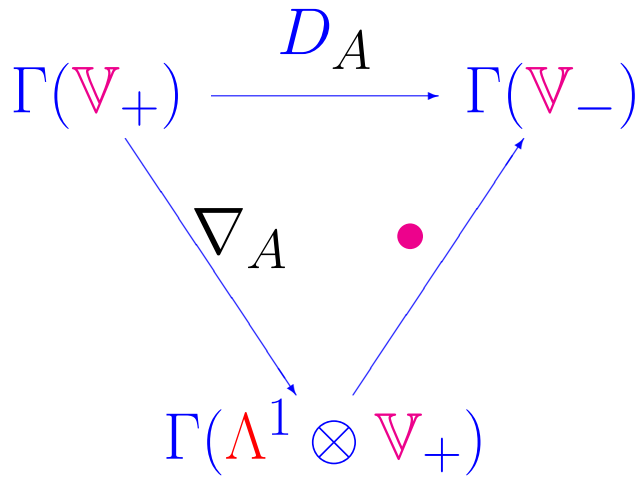

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$$\begin{aligned}\mathbb{H} &\longrightarrow \Im m\, \mathbb{H} \\ q &\longmapsto \frac{1}{2\sqrt{2}} q\, i\, \overline{q}\end{aligned}$$

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Chosen to that the SW equations imply the Weitzenböck formula

$$0 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

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$$\mathfrak{M}_{\mathfrak{c}}(g) = \{\text{solutions of SW eqs }\}/\mathscr{G}.$$

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For each metric g and spin^c structure $\mathfrak{c}$ on M, the Seiberg-Witten *moduli space* is

$$\mathfrak{M}_{\mathfrak{c}}(g) = \{\text{solutions of SW eqs }\}/\mathscr{G}.$$

Always compact, but not always a manifold. Need to overcome in order to define Witten's invariant.

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If M simply connected, just constant maps S^1.

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In general,

$$\mathscr{G}_0 \cong S^1 \rtimes H^1(M, \mathbb{Z}).$$

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Can avoid these via the following trick...

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When $\eta = 0$, becomes standard SW system.

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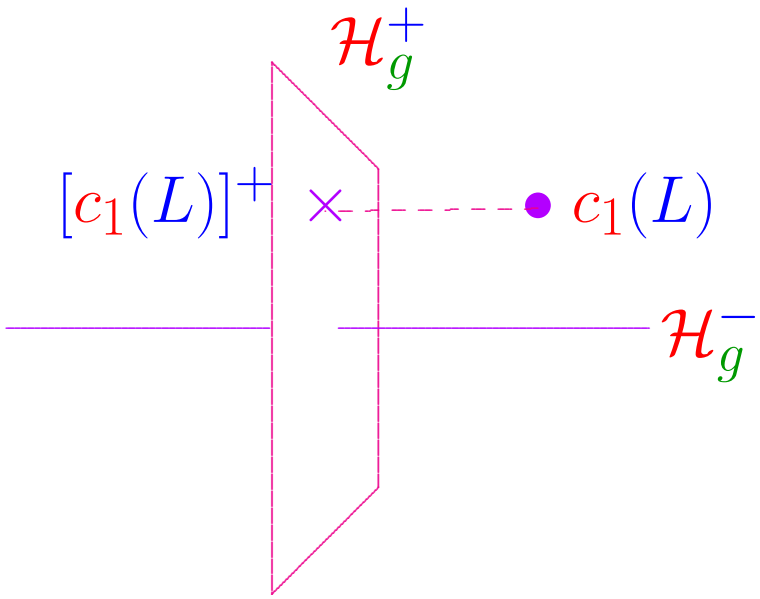

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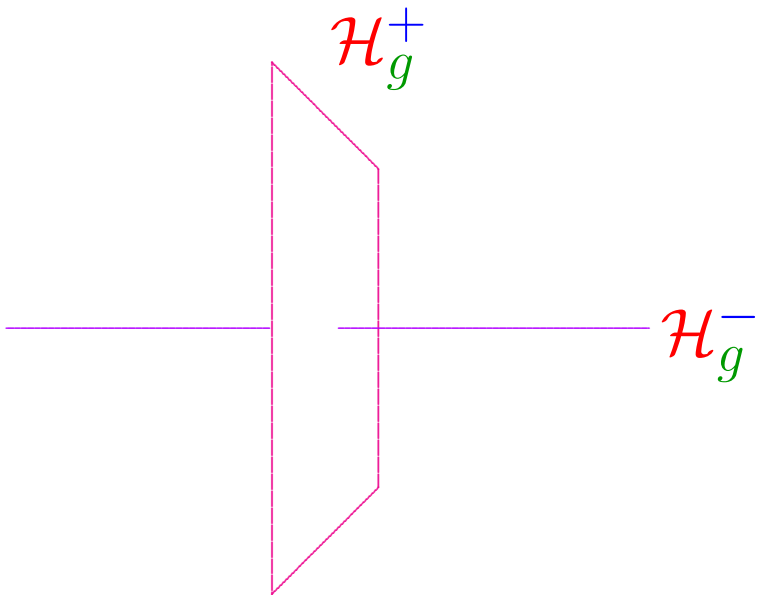

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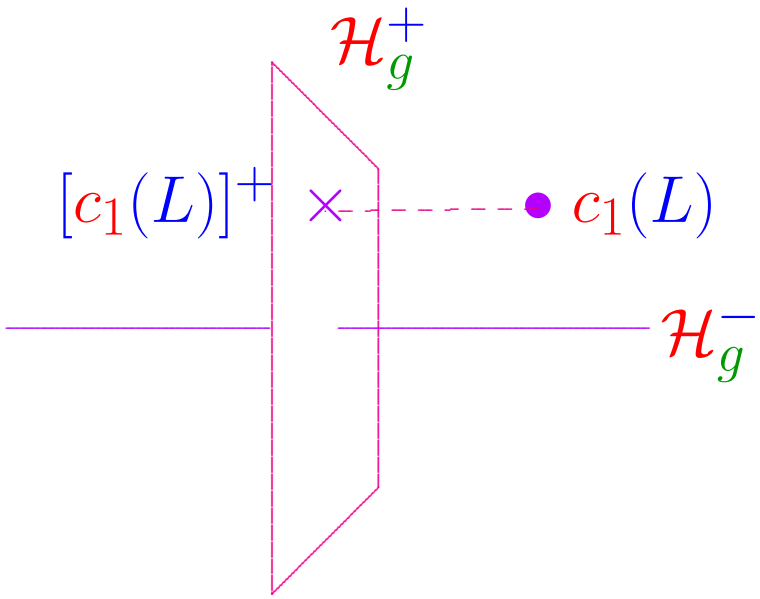

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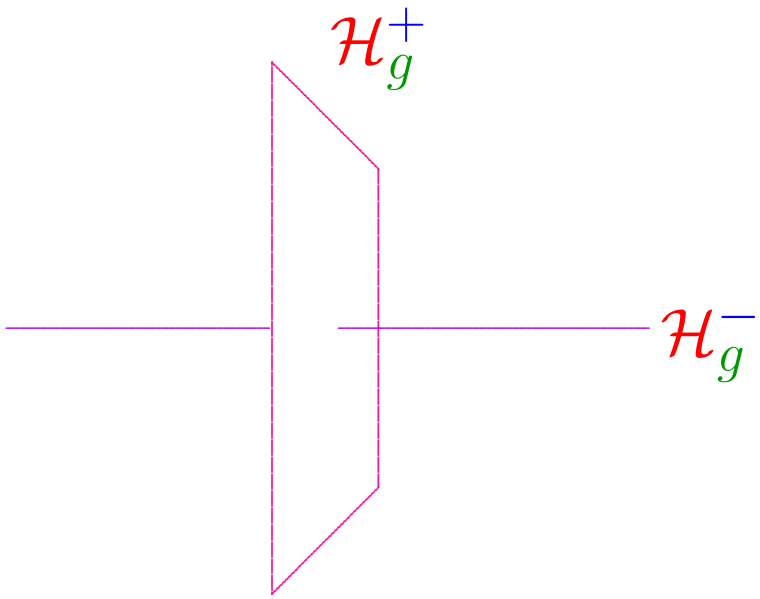

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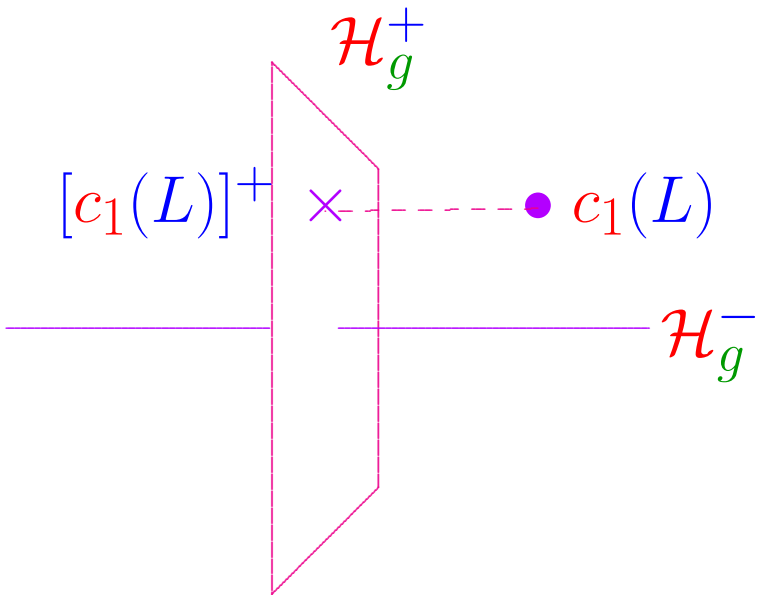

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(Linearization surjective at every preimage.)

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Dimension = index of linearization:

$$\dim = \frac{{c_1}^2(L) - (2\chi + 3\tau)(M)}{4} + 1.$$

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$$\dim \mathfrak{M}_{\mathfrak{c}}(g, \eta) = \frac{{c_1}^2(L) - (2\chi + 3\tau)(M)}{4}.$$

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images of closed bounded sets are compact!

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Simplest case: ${c_1}^2(L) = (2\chi + 3\tau)(M)$.

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Simplest case: ${c_1}^2(L) = (2\chi + 3\tau)(M)$.

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Every unitary connection A on L induces spin^c Dirac operator

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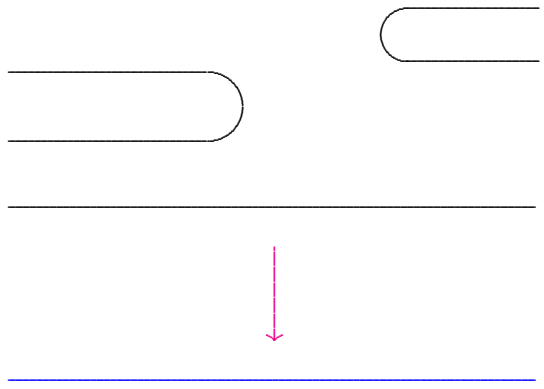
Any two good choices (g, η) can be a joined by a smooth curve transverse to the monopole map.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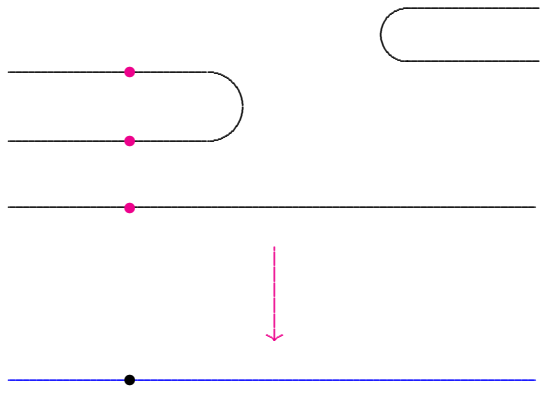
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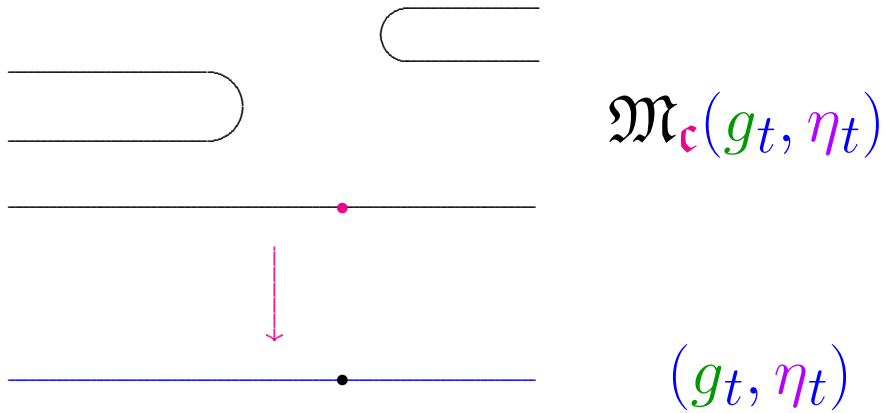
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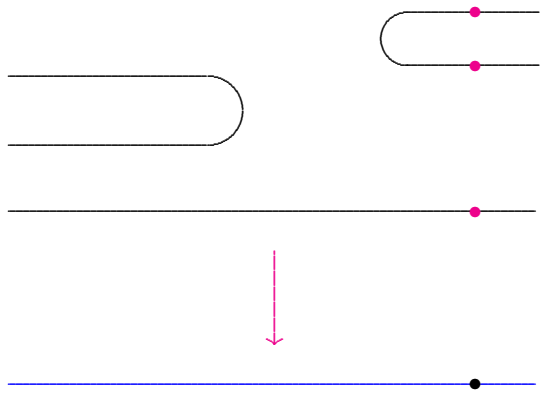
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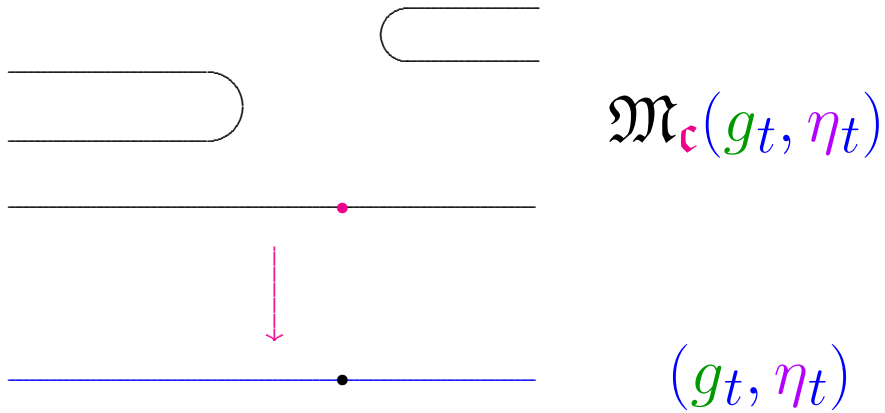
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There is actually a $\mathbb{Z}$-valued improvement, but we will not need this for our purposes here.

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Linearization surjective at this unique solution, so $n_{\mathfrak{c}}(M) = 1 \bmod 2$.

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Theorem (Taubes)**.** *Let (M, ω) be a smooth compact symplectic manifold with $b_+(M) \geq 2$. Then $c_1(M, \omega)$ is basic class of M.*

Proof similar: there are choices of η for which there is a unique SW solution, up to gauge equivalence.

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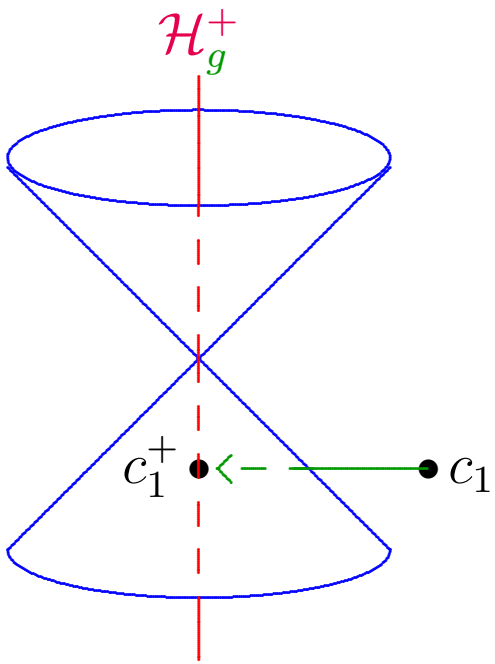
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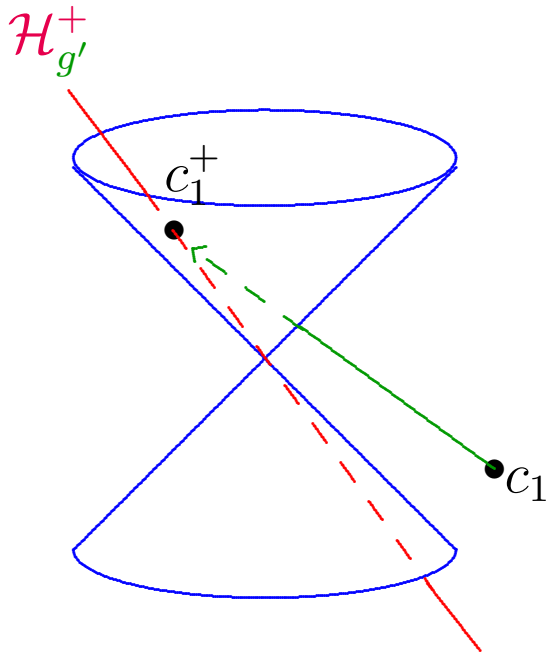
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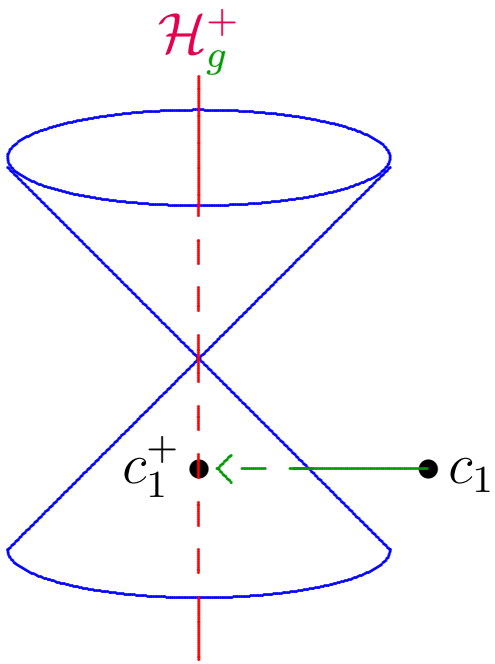
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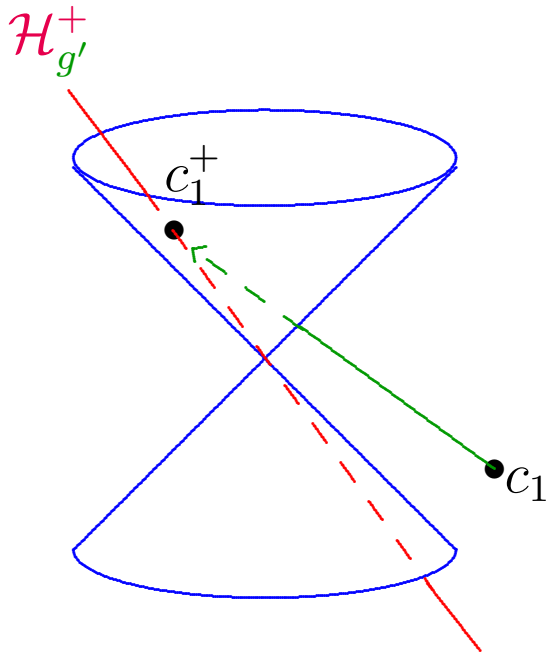
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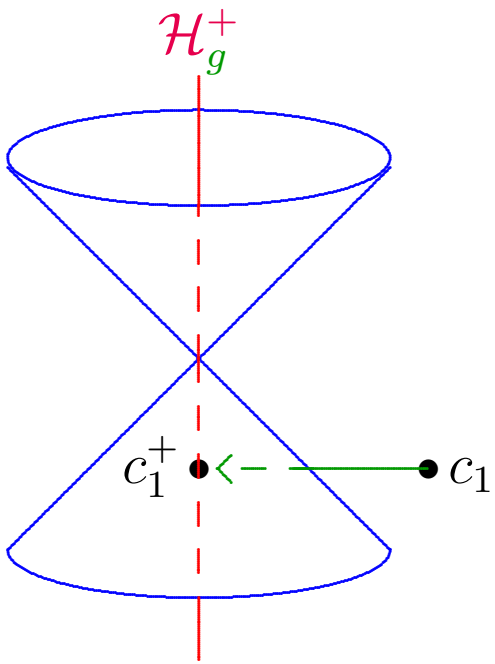
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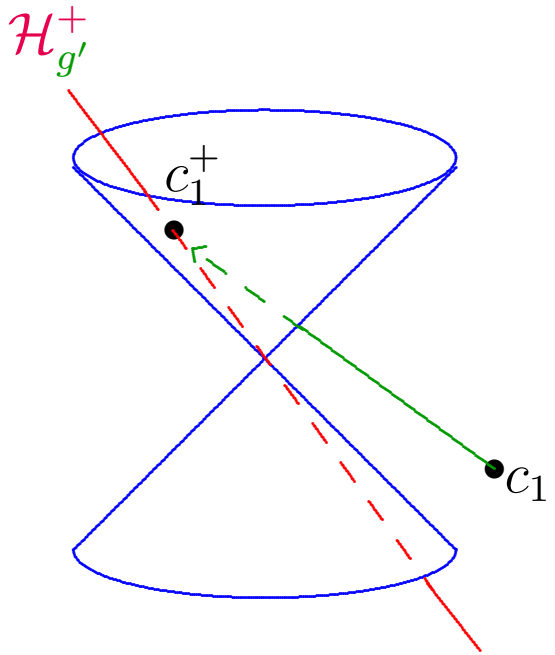
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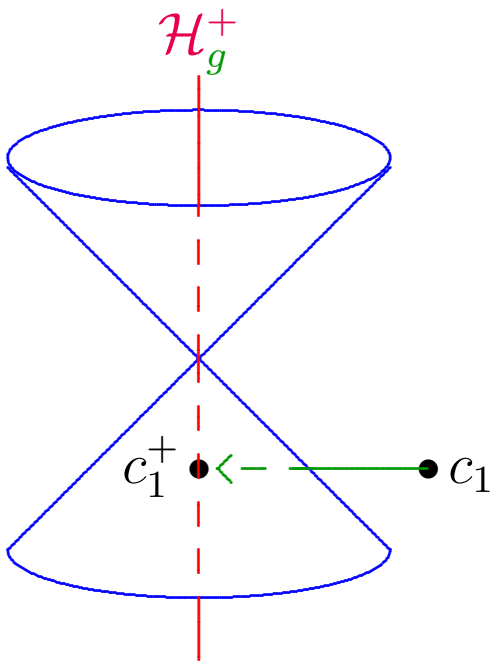
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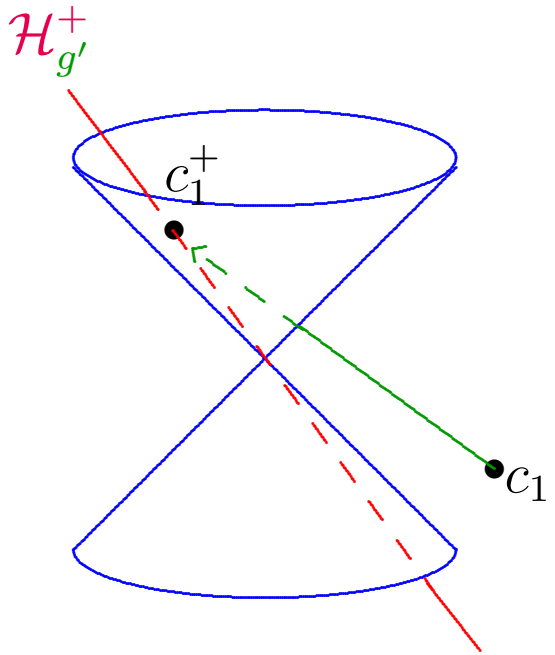
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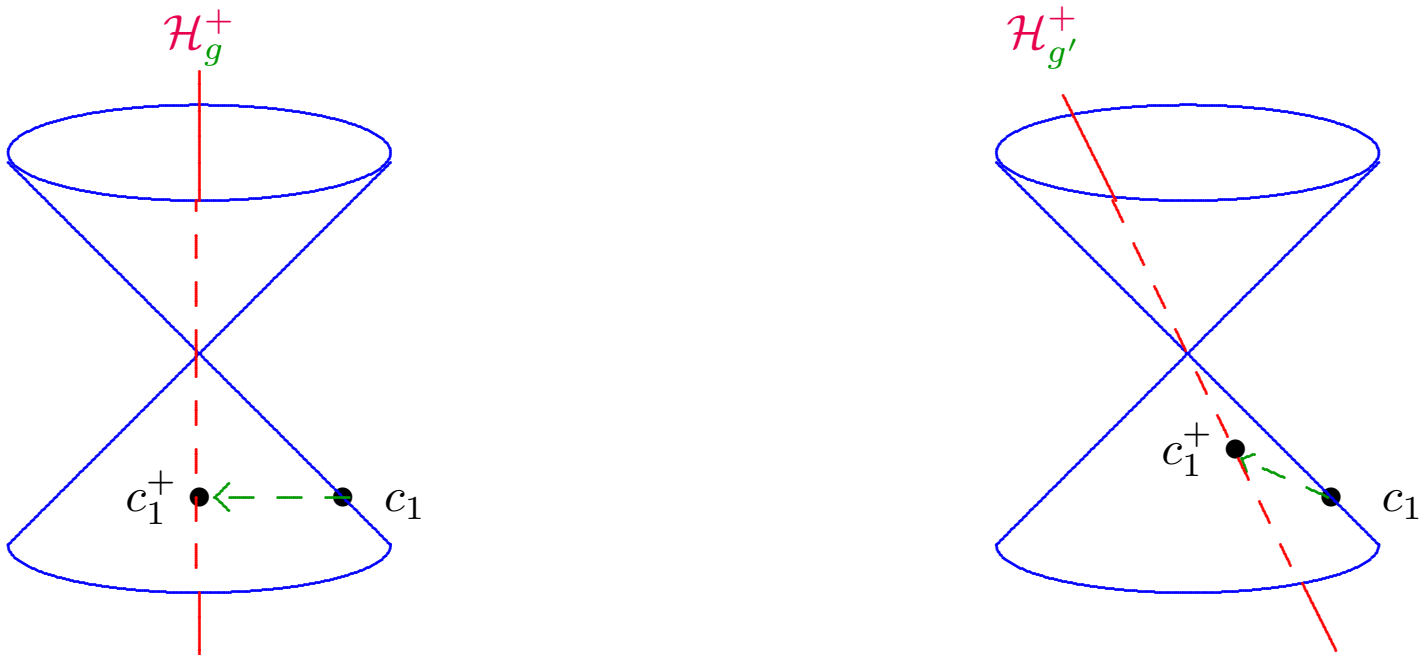
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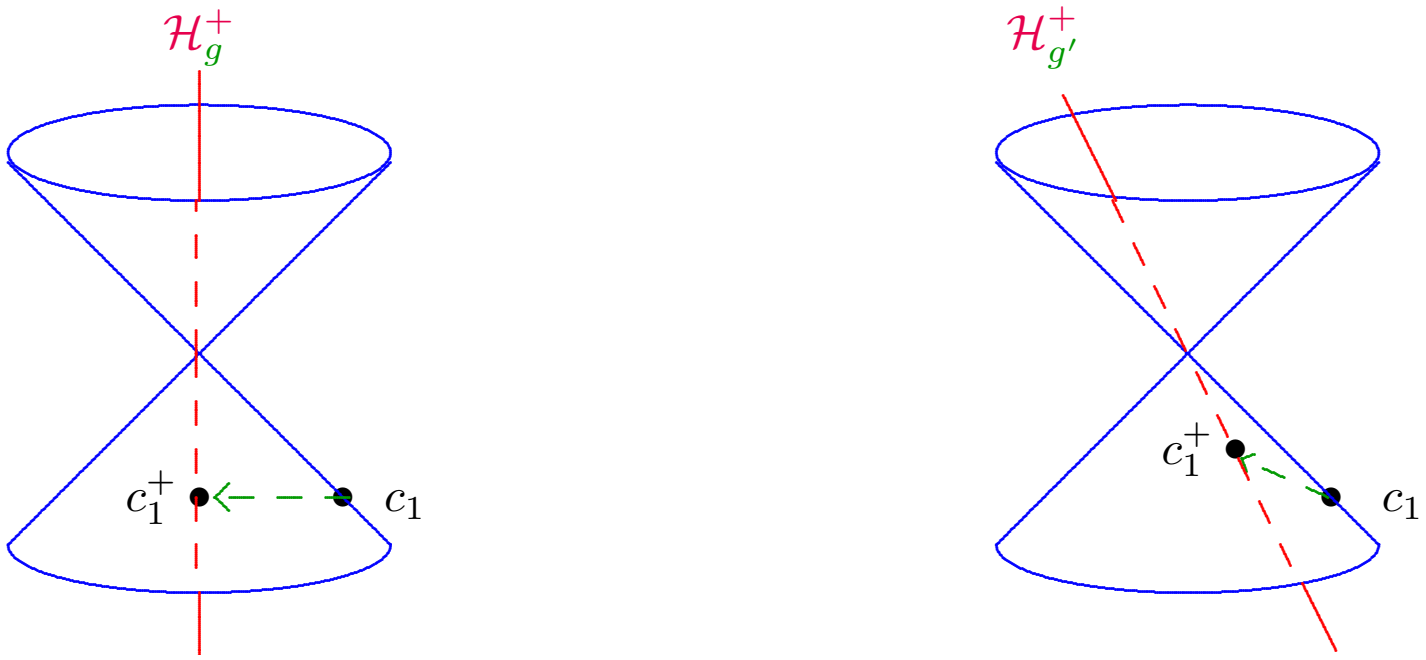
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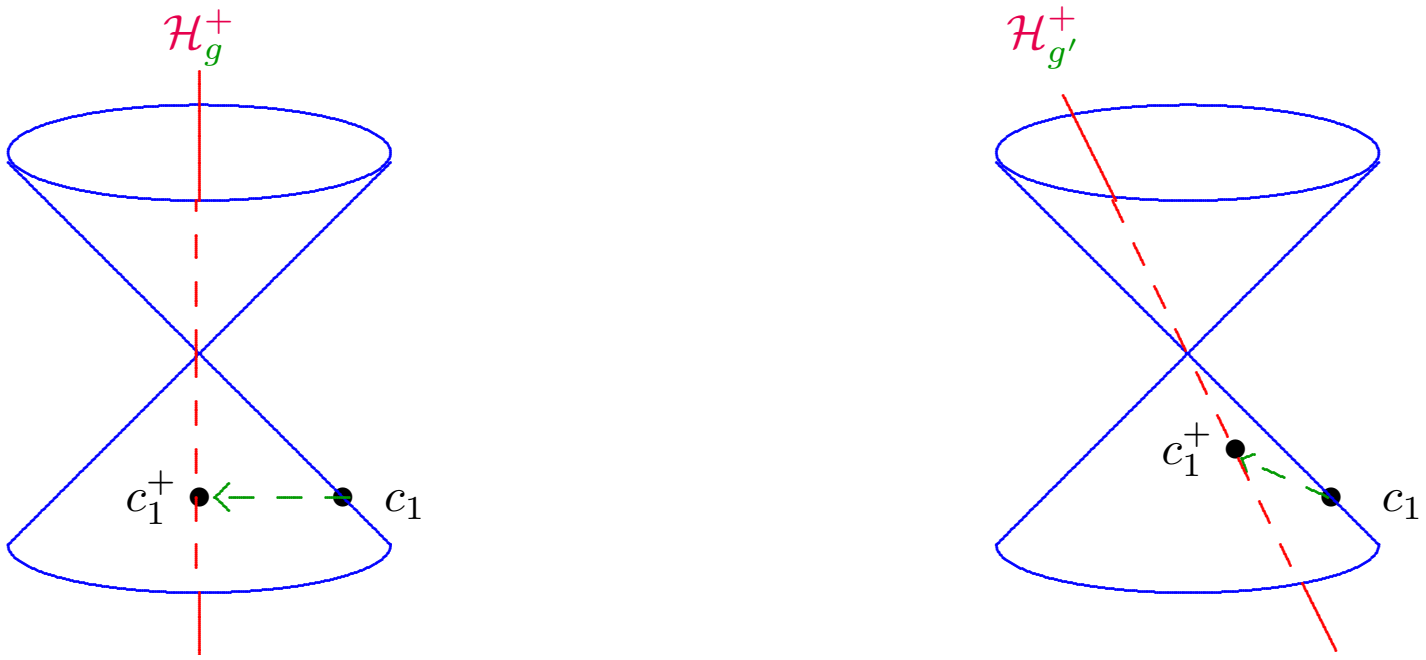
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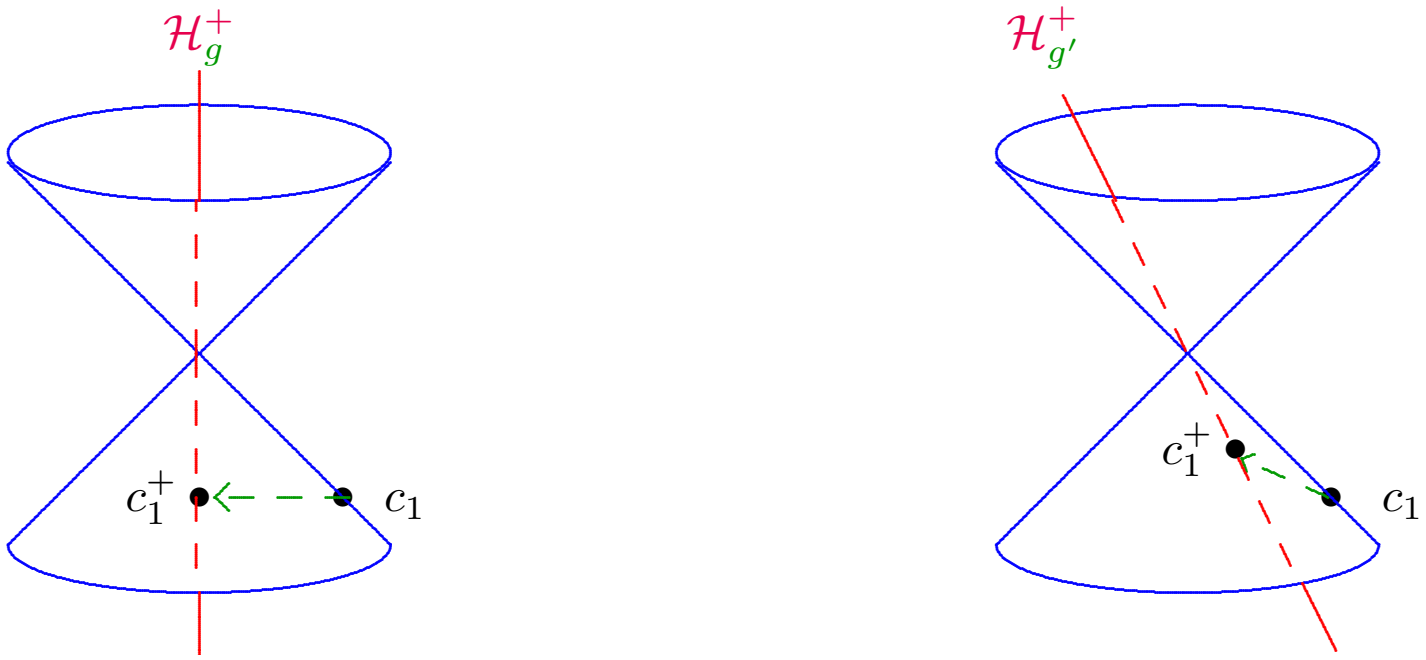
Concretely, this condition will automatically hold whenever $\max |\eta|$ is sufficiently small.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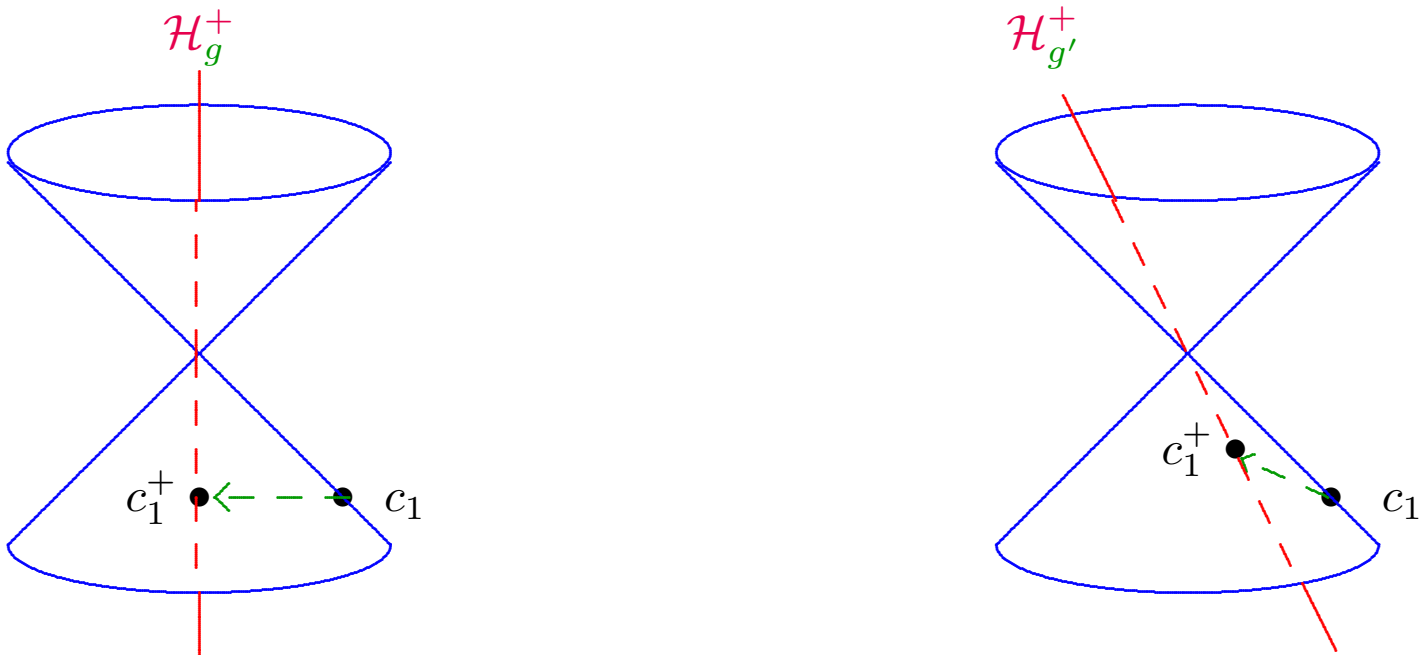
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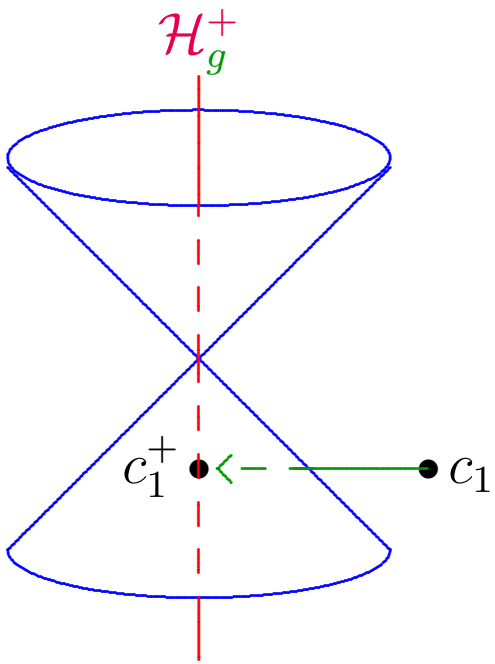
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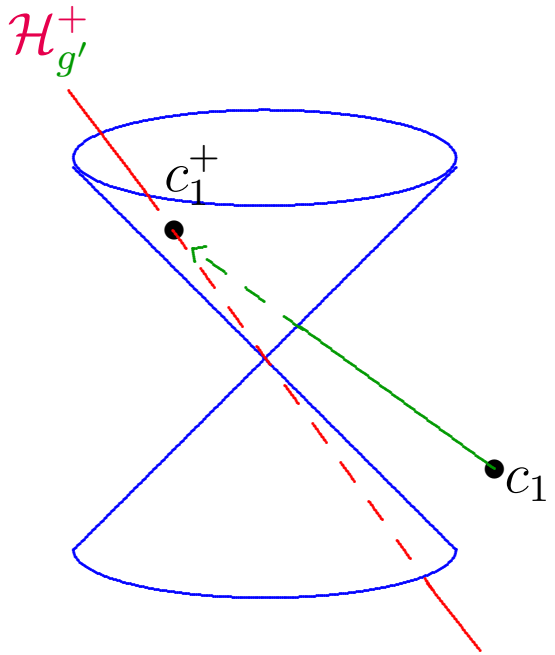
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We now return to the simpler setting of $b_+ \geq 2 \ldots$

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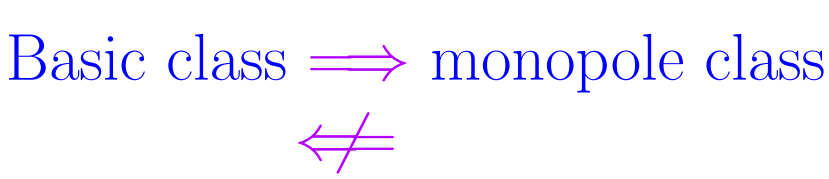
$$c_1{}^{\mathbb{R}}(L) = c_1{}^{\mathbb{R}}(M, \det \mathbb{V}_+) \in H^2(M, \mathbb{Z})/(\textit{torsion})$$

is a monopole class of M.

Basic class $\Longrightarrow$ monopole class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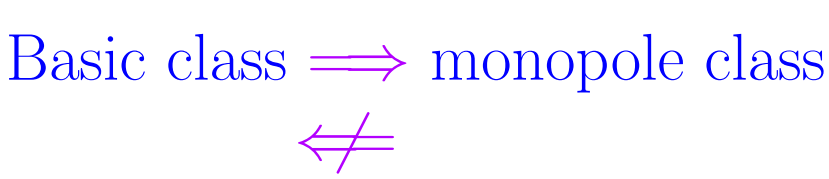
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Corollary. *Let M be a smooth compact manifold with $b_+(M) \geq 2$. Then the collection*

$$\mathfrak{S} \subset H^2(M, \mathbb{R})$$

of all basic classes is finite.

Intermission

Witten:

consider both Φ and A as unknowns,

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Non-linear, but elliptic once 'gauge-fixing'

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

SW equations imply the Weitzenböck formula

$$0 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

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SW invariant $\in \mathbb{Z}_2$ like mod-2 mapping degree.

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Implies non-existence of metrics g for which $s > 0$.

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with equality only if (M, g) is locally hyper-Kähler. The latter case happens only if M finitely covered by flat T^4 or $K3$.

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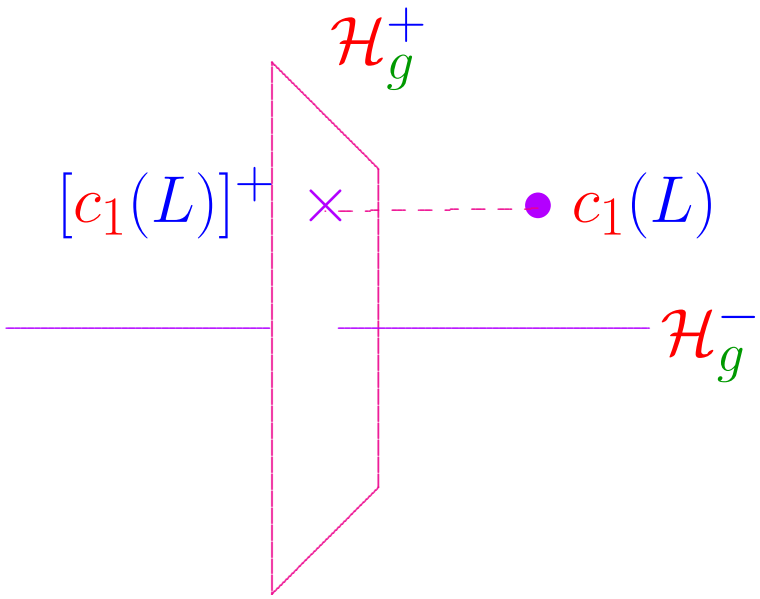
$$c_1(L) \in H^2(M, \mathbb{R}) = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$$



$H^2(M, \mathbb{R})$

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Moreover, A is Chern connection on $L = K^{-1}$.

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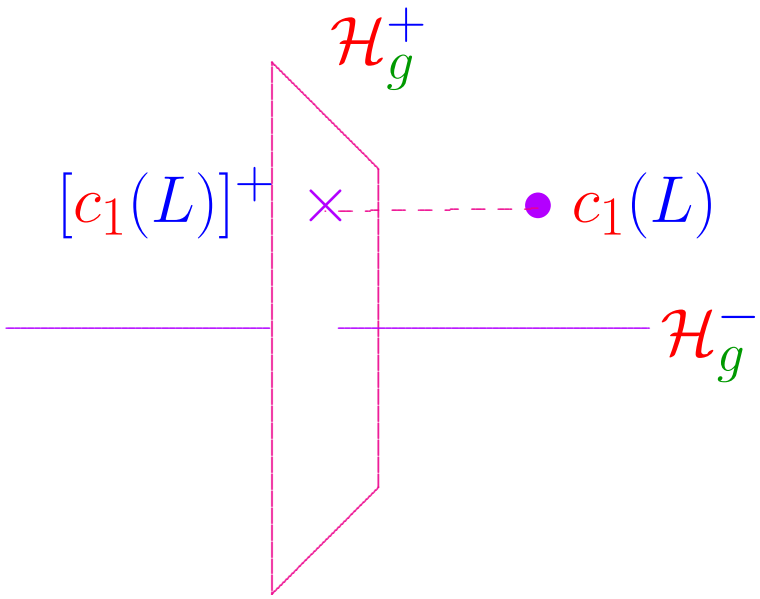
We need metric-independent improvement!



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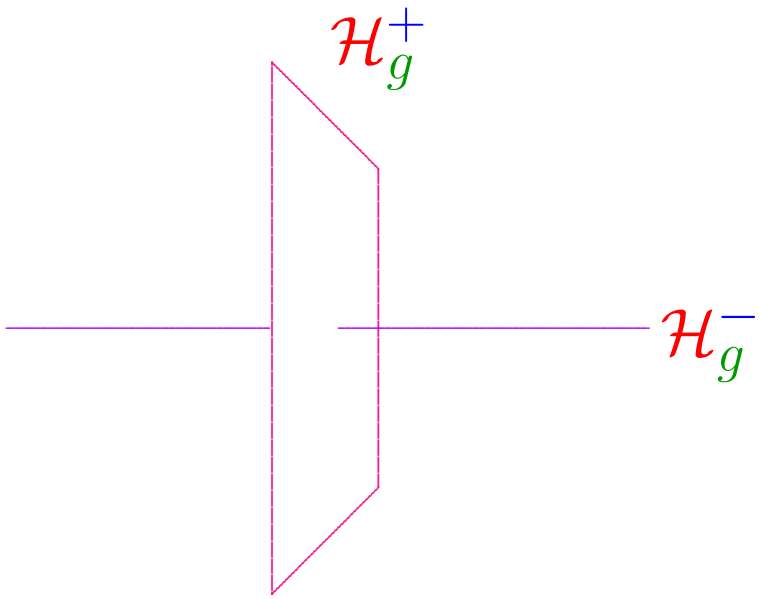

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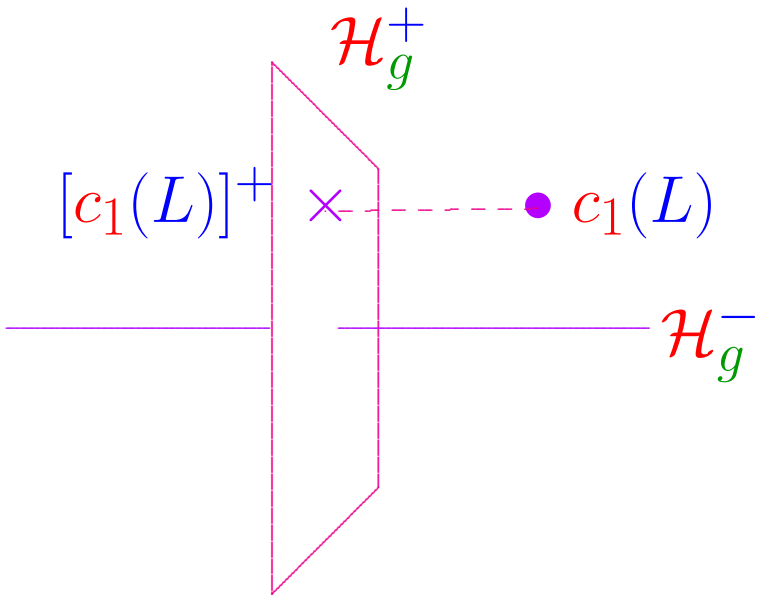

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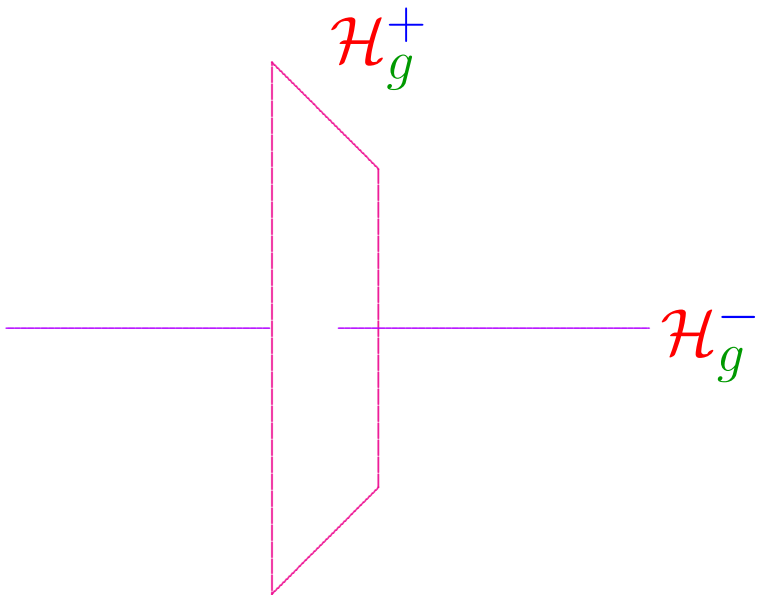

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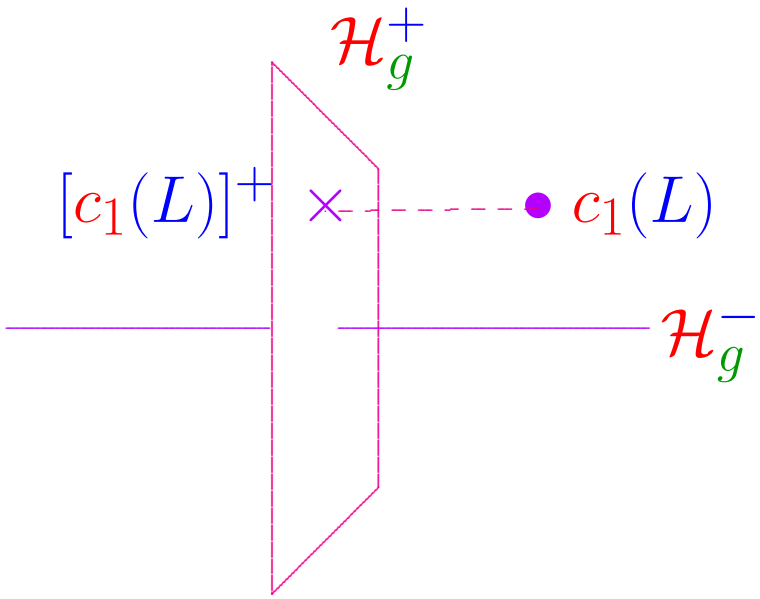

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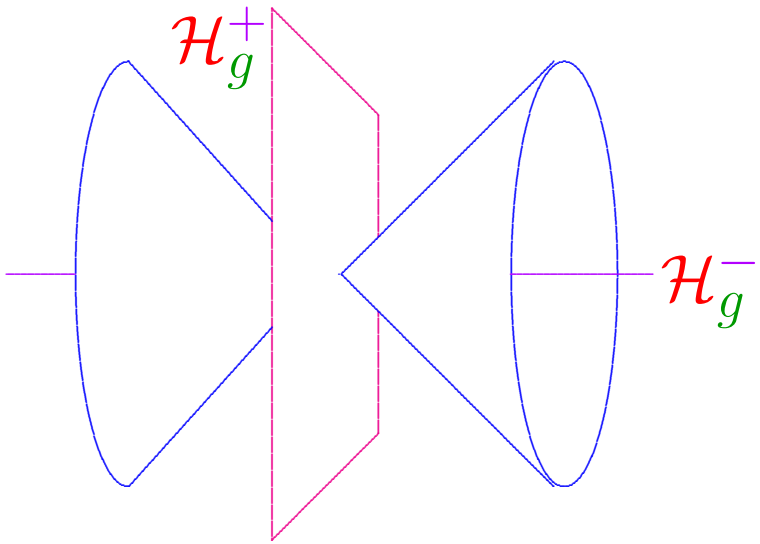

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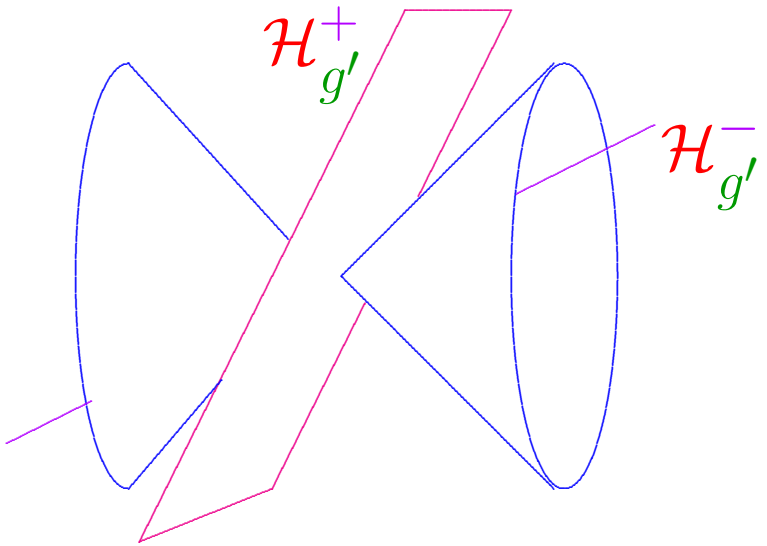

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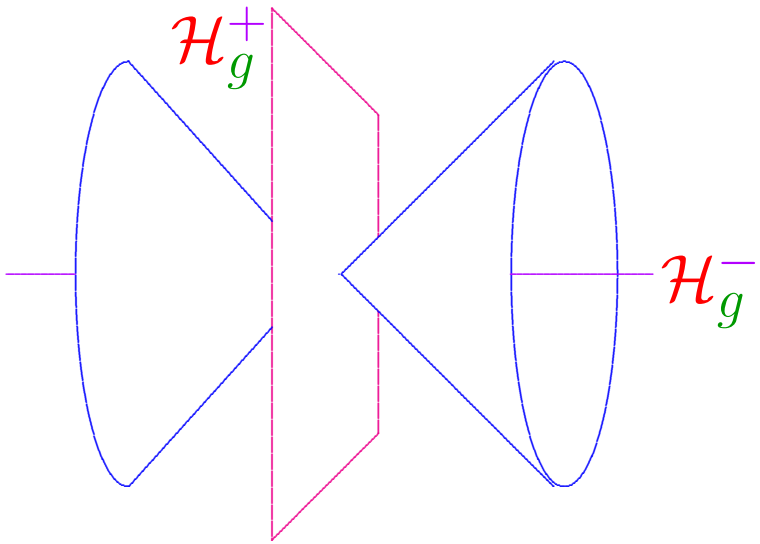

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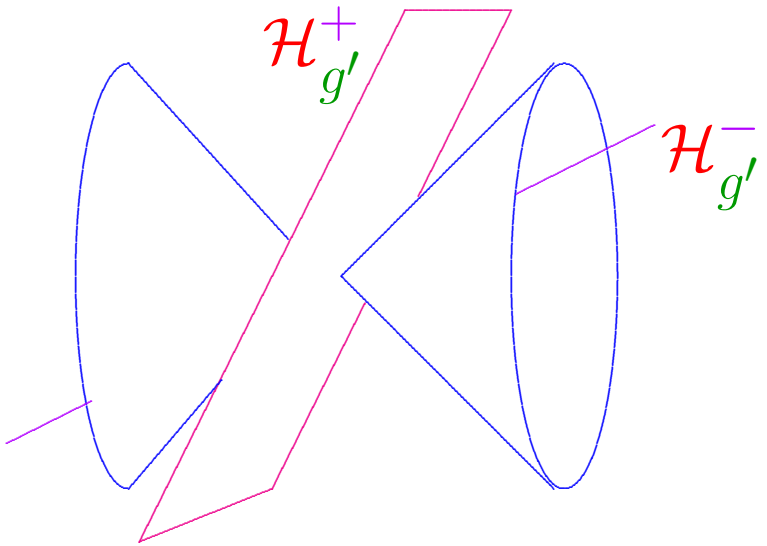

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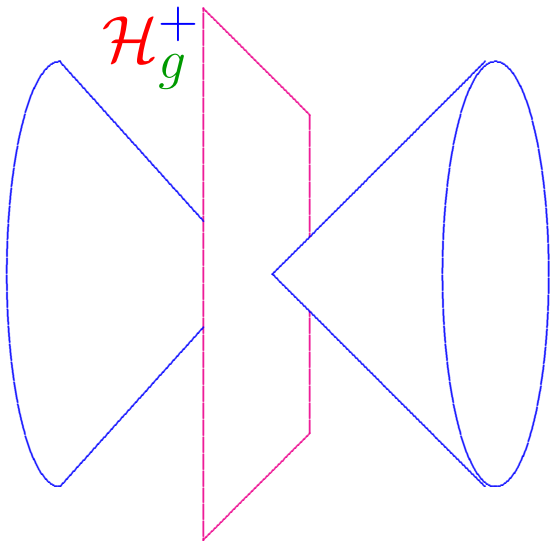

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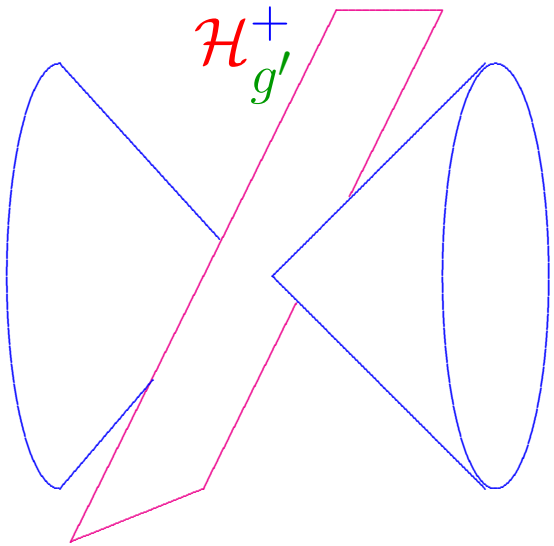

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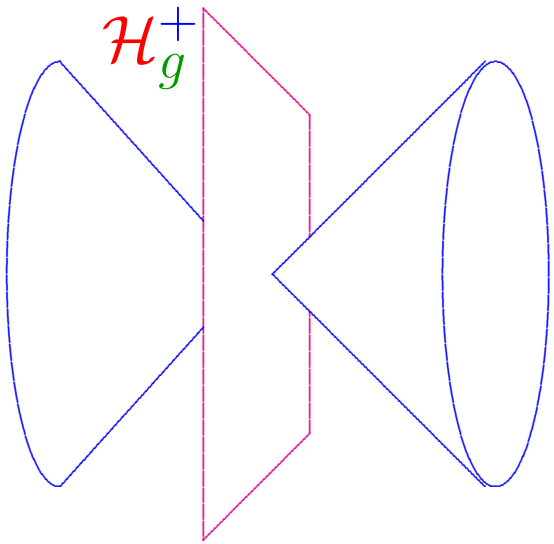

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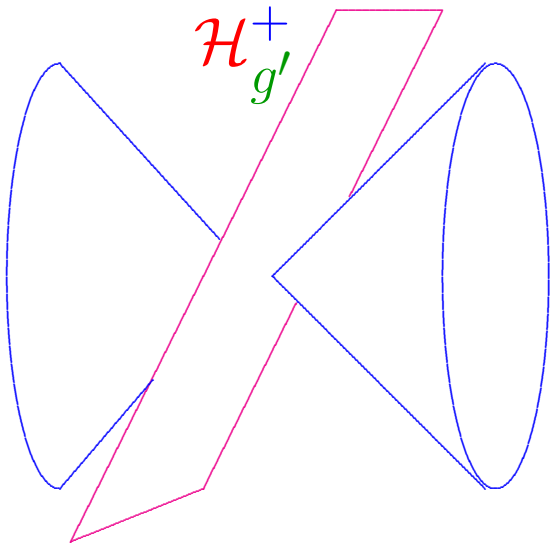

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will be called a basic class of M.

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have a solution (Φ, A) for every metric g on M.

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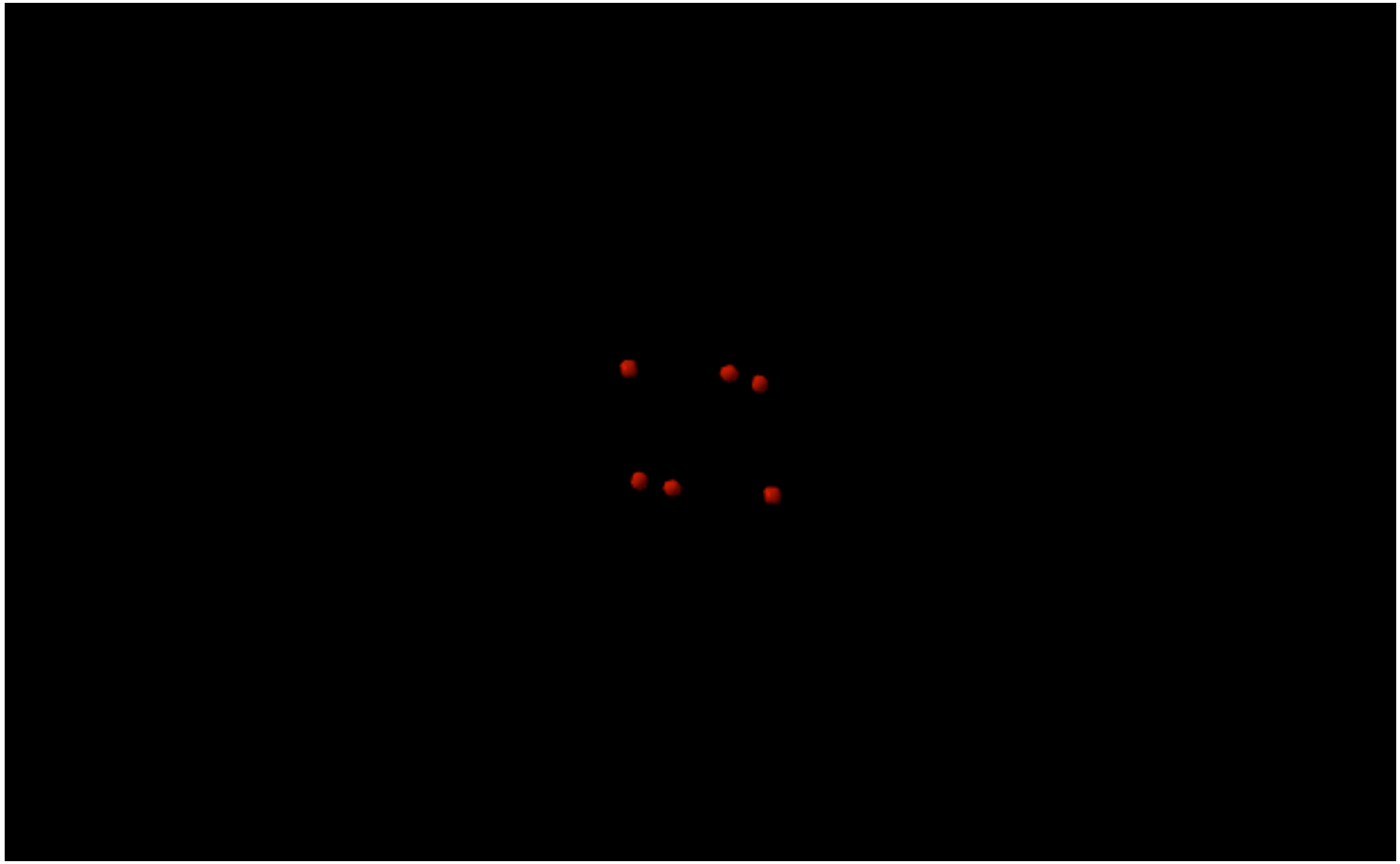

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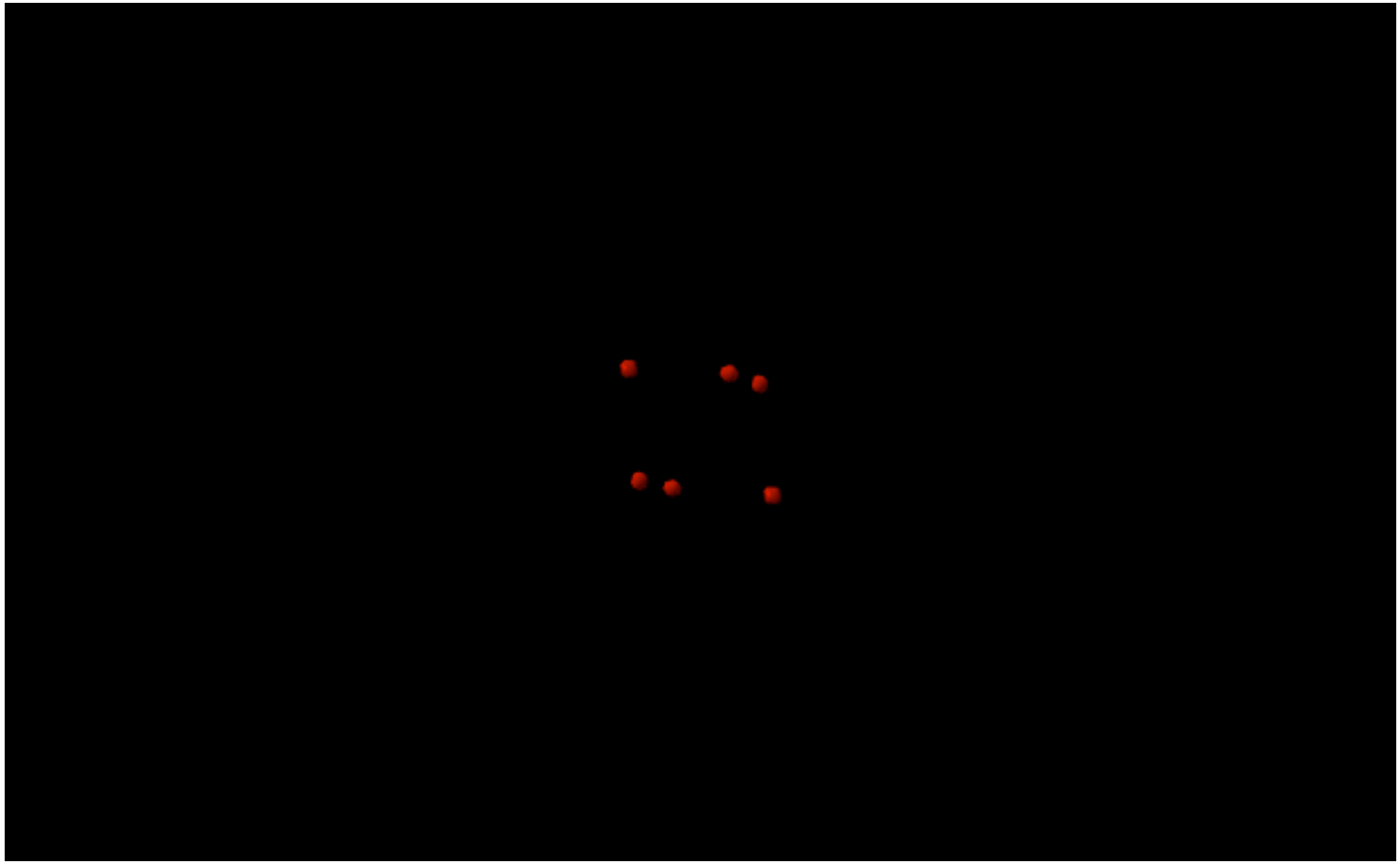

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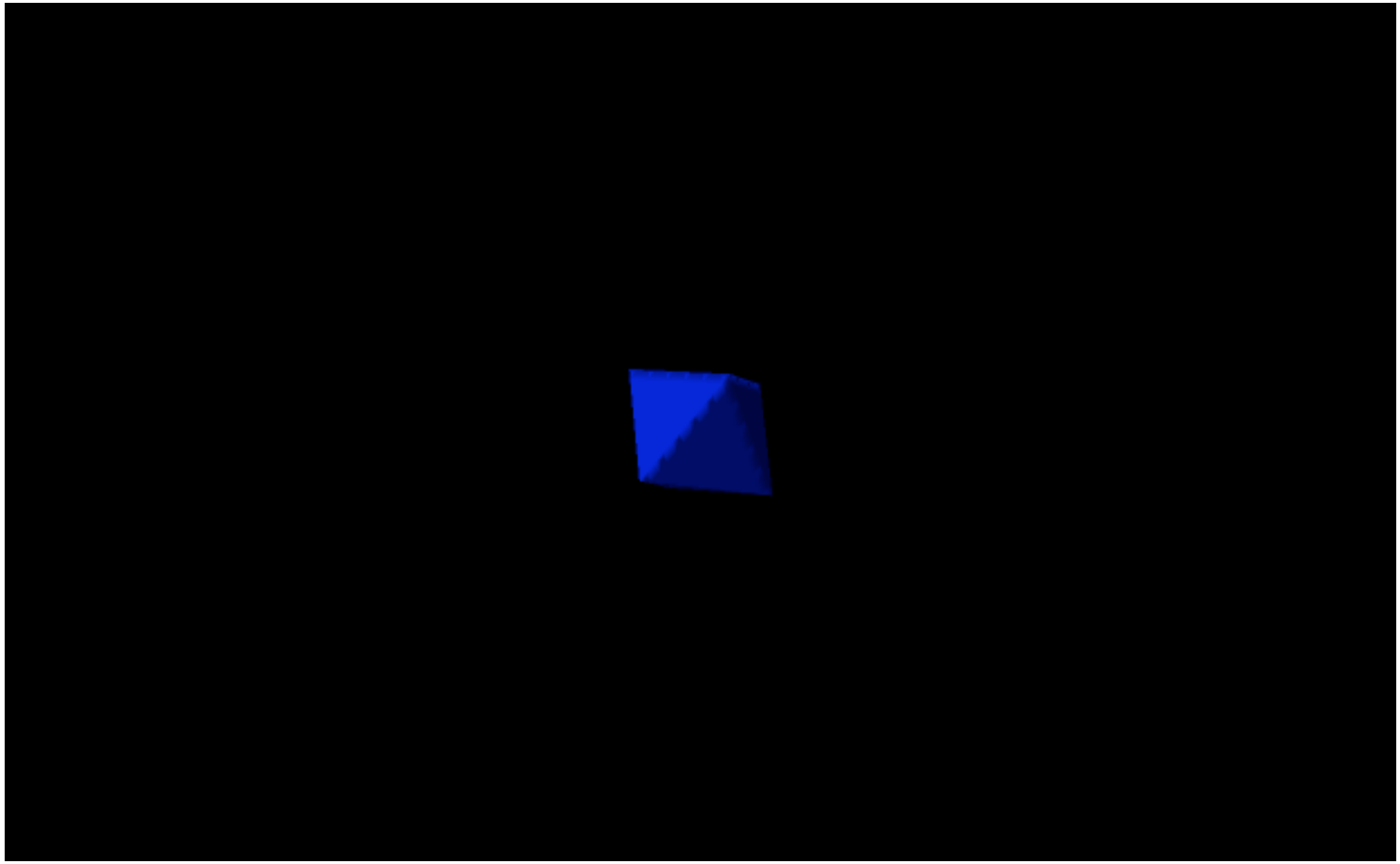

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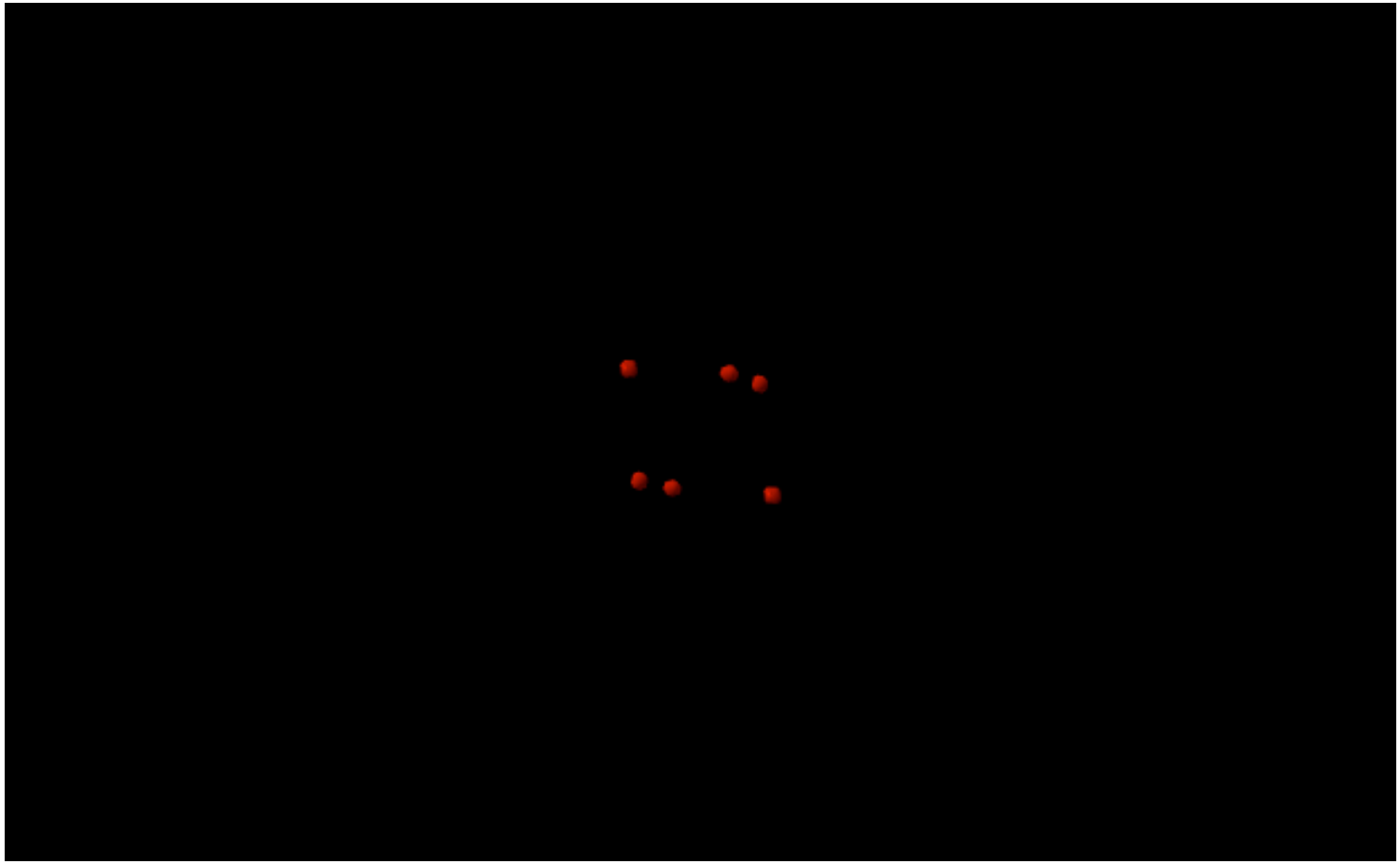

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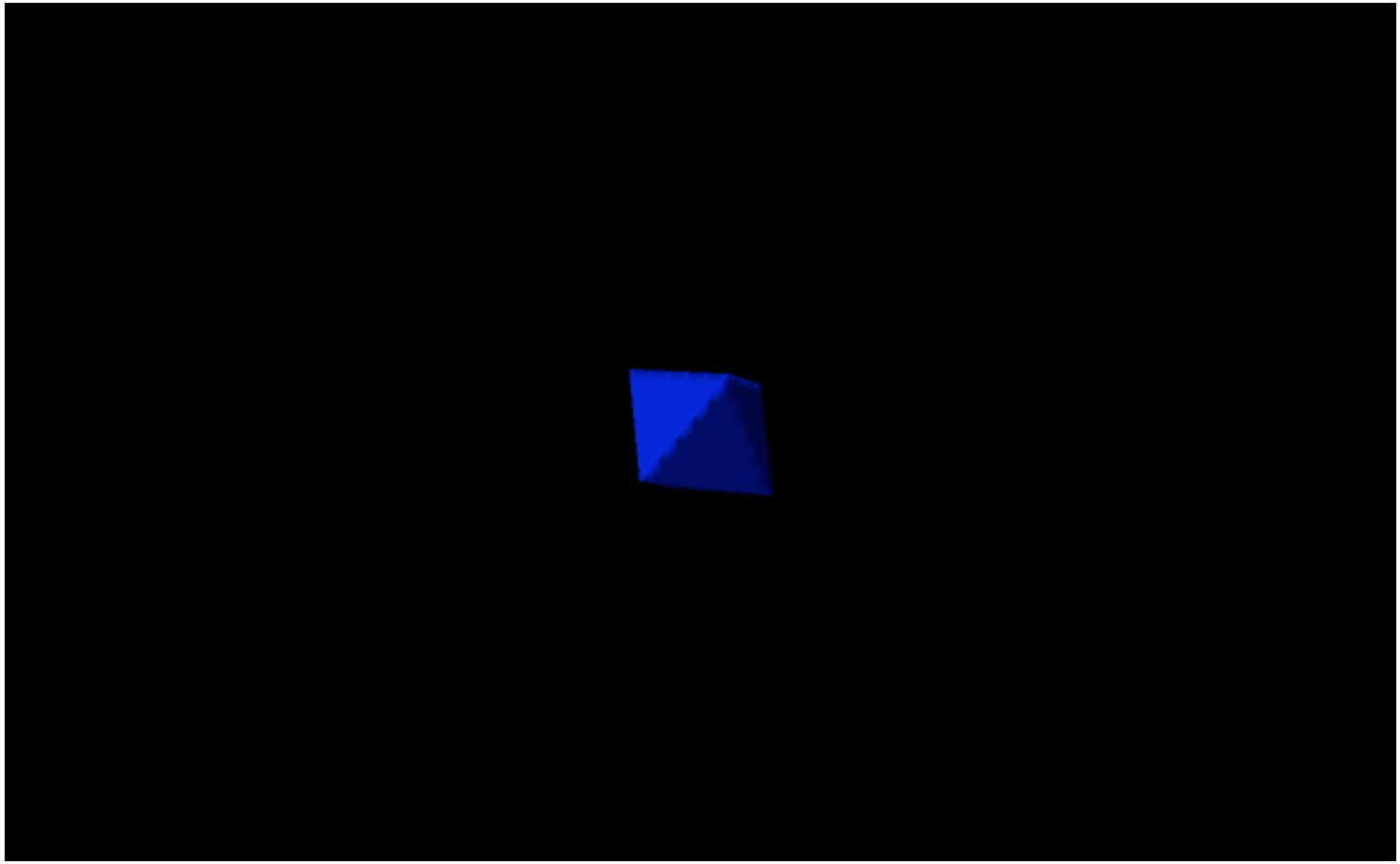

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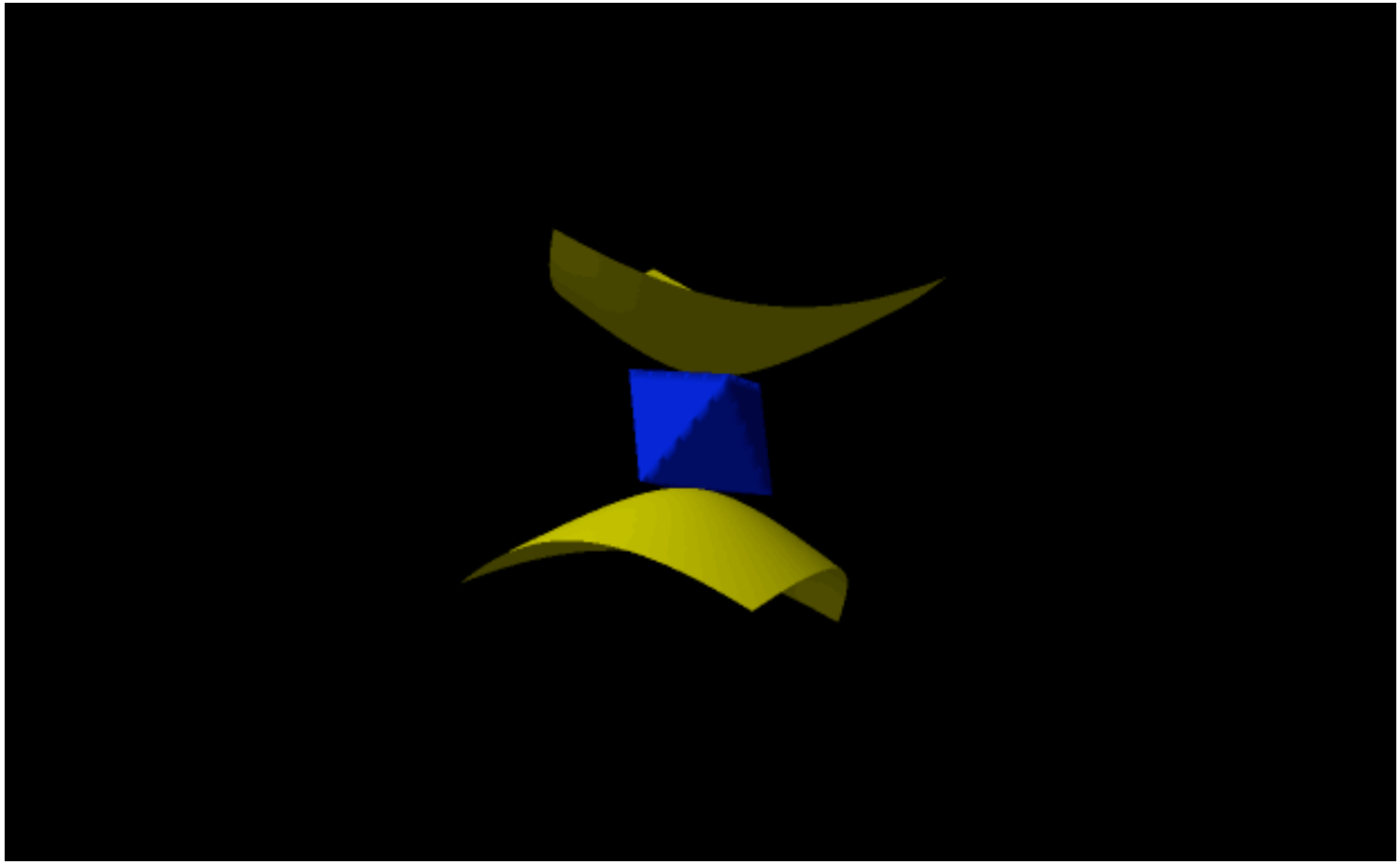

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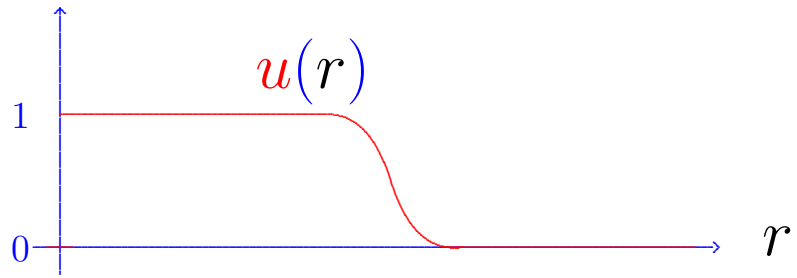

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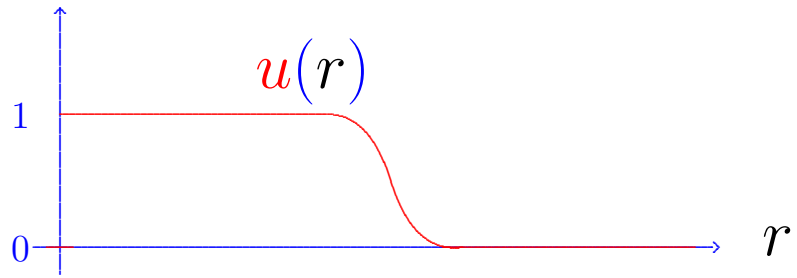

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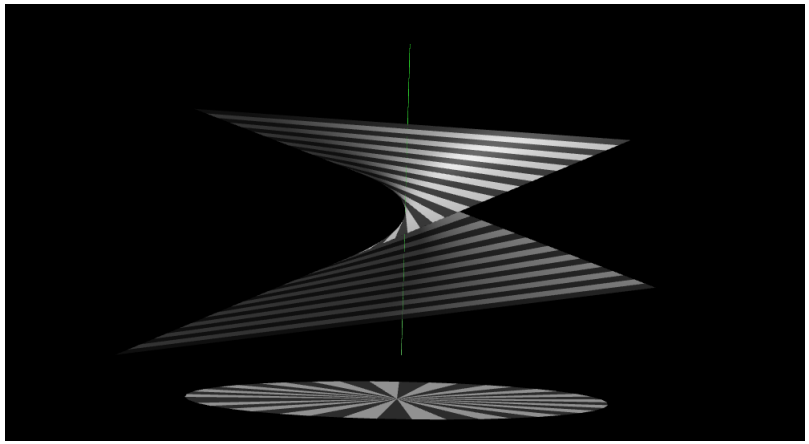

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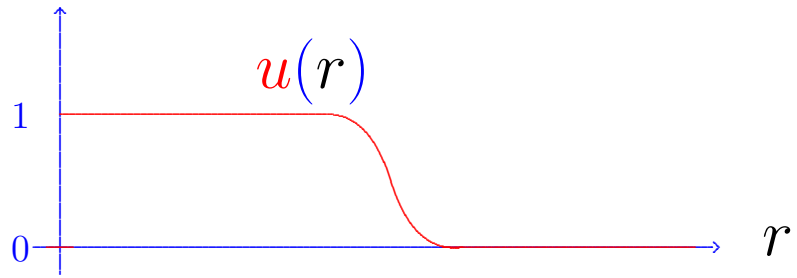

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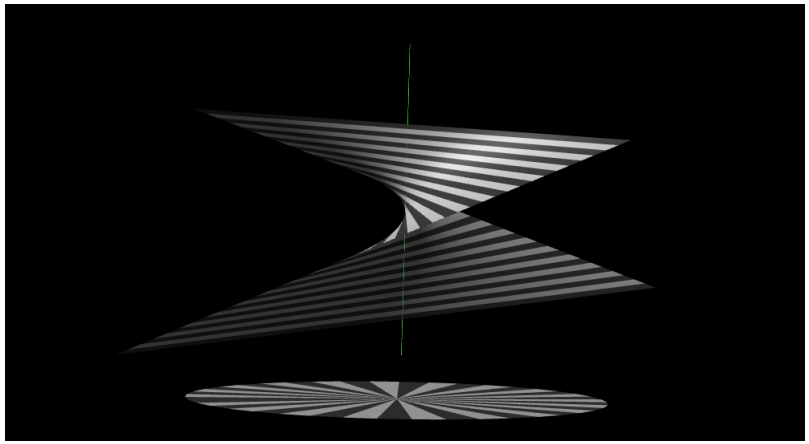

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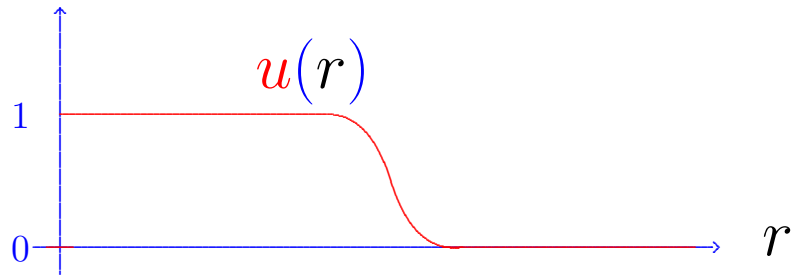

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Moreover, equality holds in either case iff $M = X$, and g is Kähler-Einstein with $\lambda < 0$.

Cauchy-Schwarz argument $\Longrightarrow$

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Theorem. *Let X be a minimal surface of general type,*

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So being "very" non-minimal is an obstruction.

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When $n = 2m = 4$, such M are the minimal complex surfaces of general type such that

$$\nexists \mathbb{CP}_1 \overset{\mathcal{O}}{\hookrightarrow} M$$

of homological self-intersection -2.

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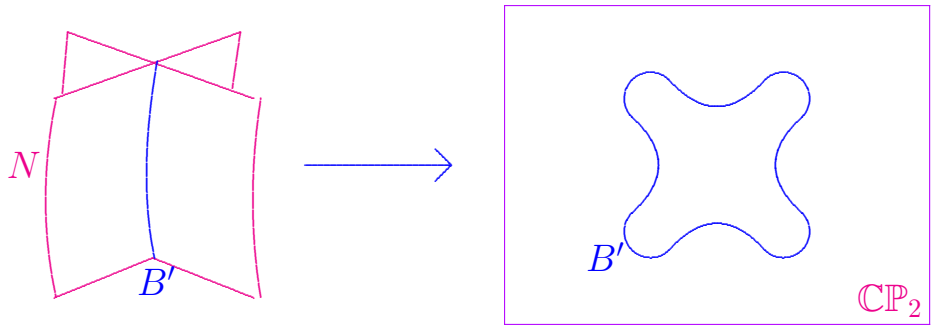

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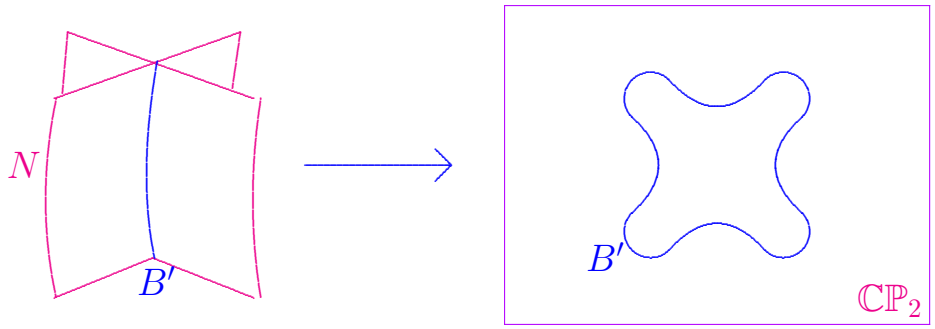
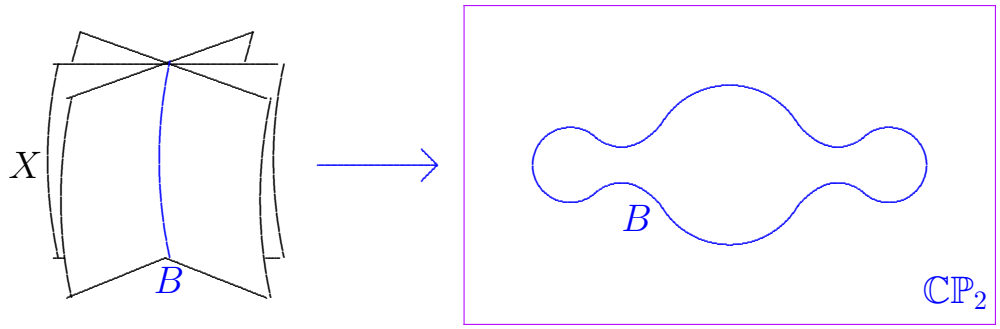


Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



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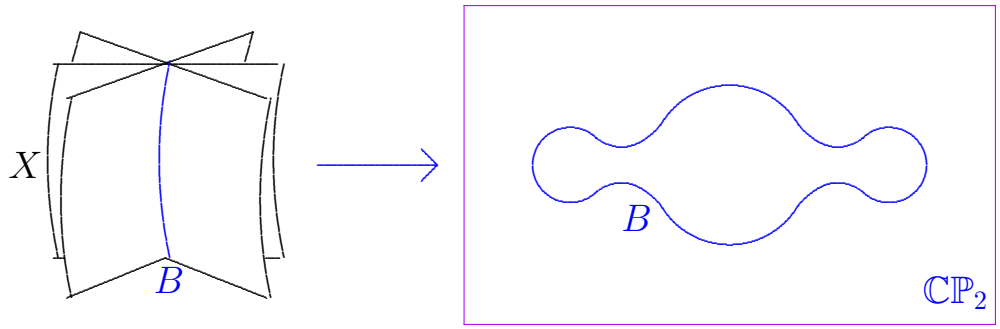

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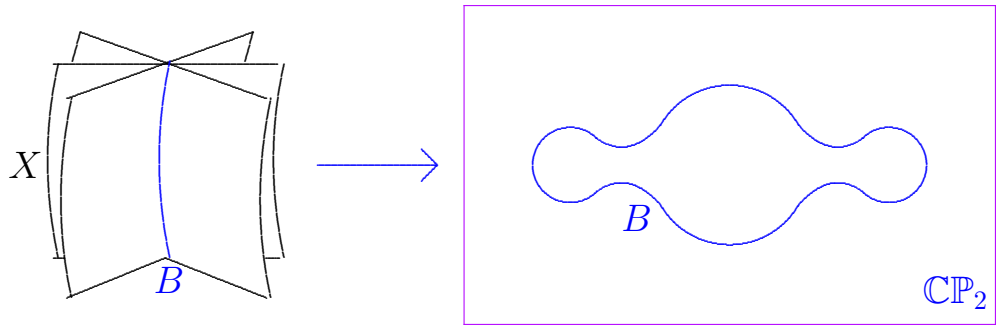

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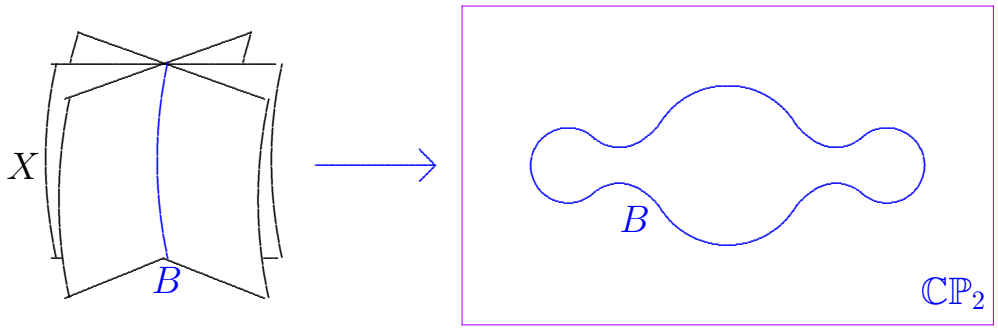

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Theorem $\Longrightarrow$ *no* Einstein metric on M.

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Moral: Existence depends on diffeotype!

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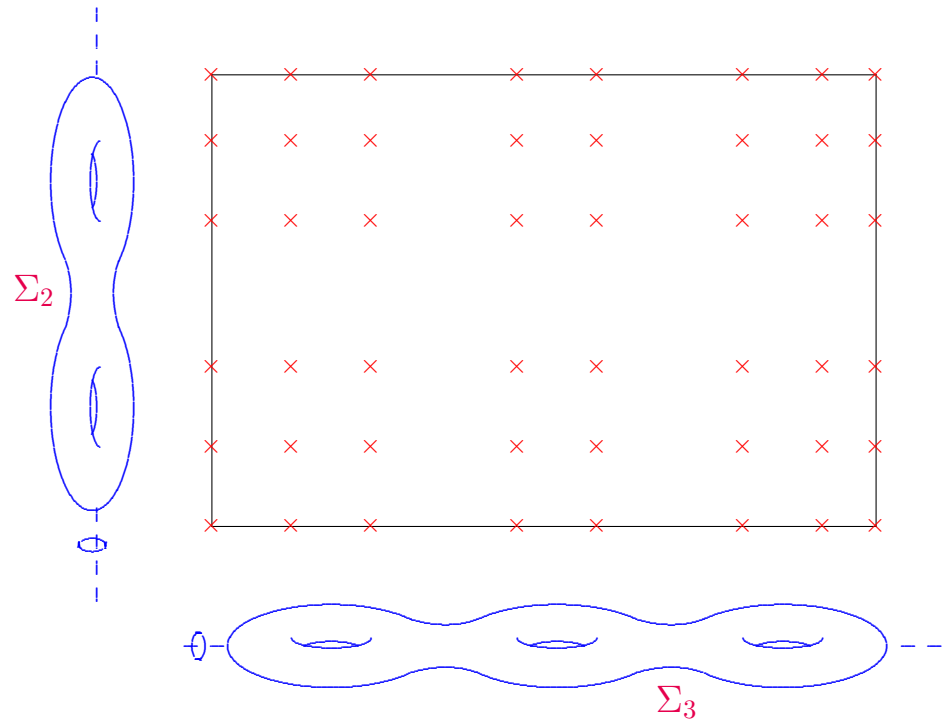

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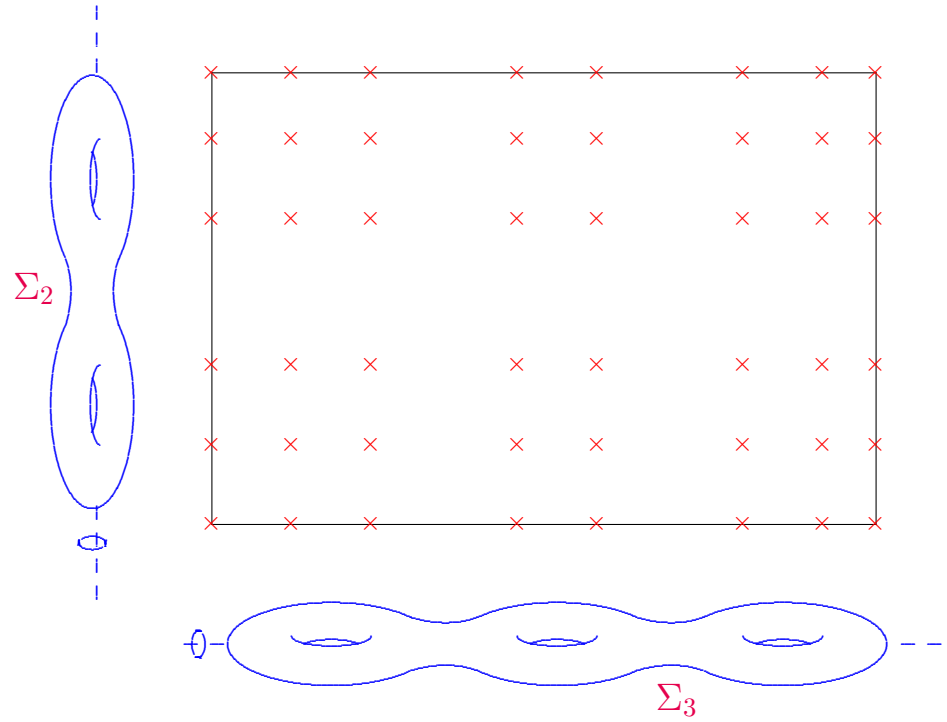

Obtain N by replacing nbhds of 48 singularities of

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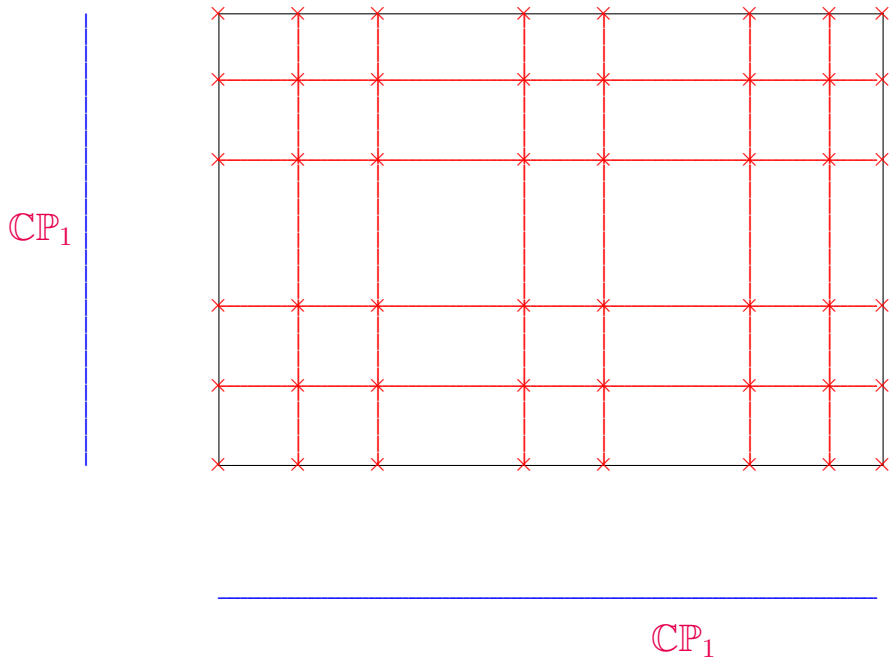

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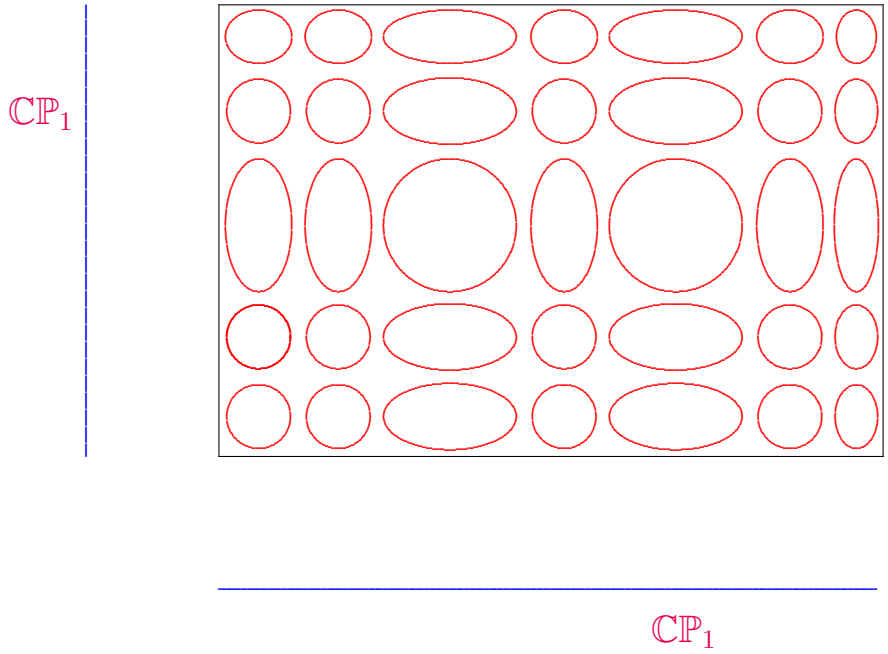

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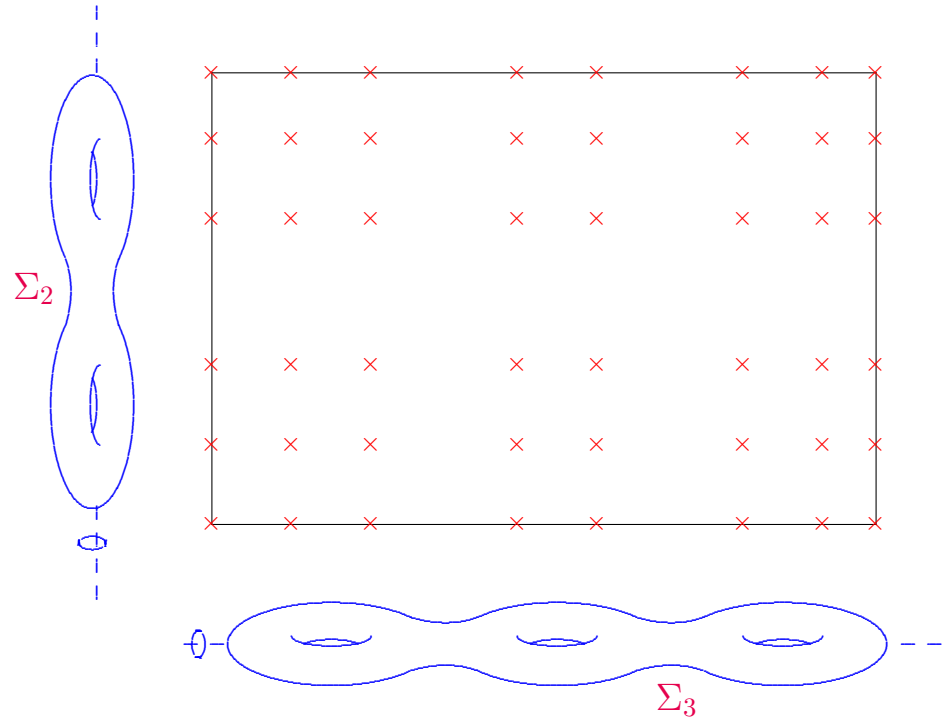

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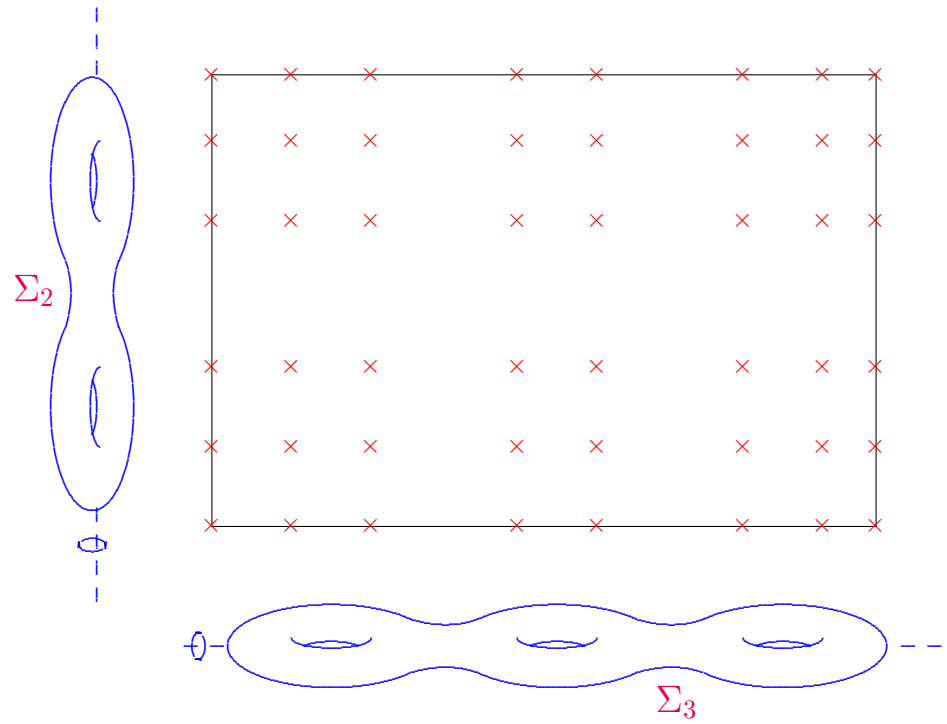

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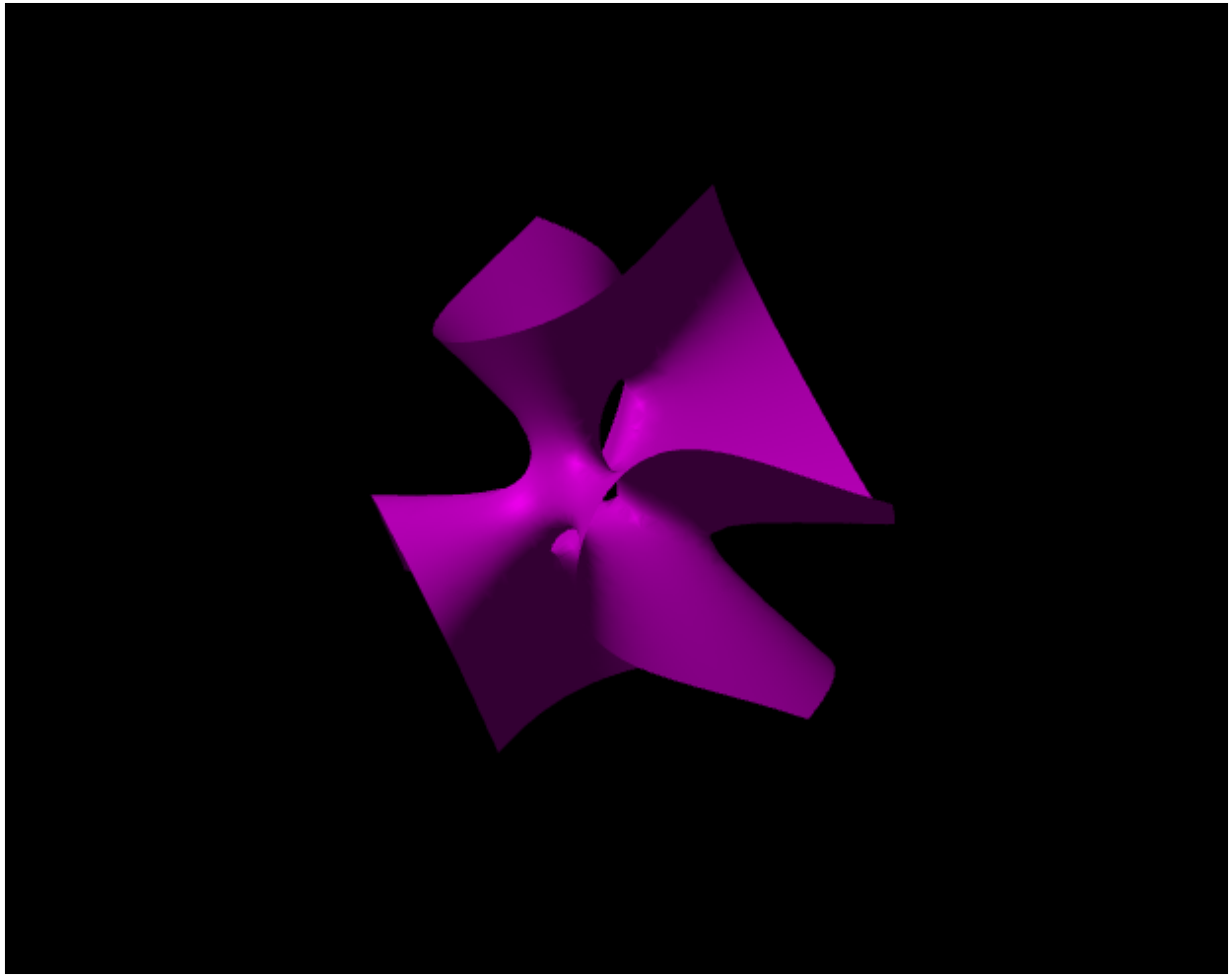

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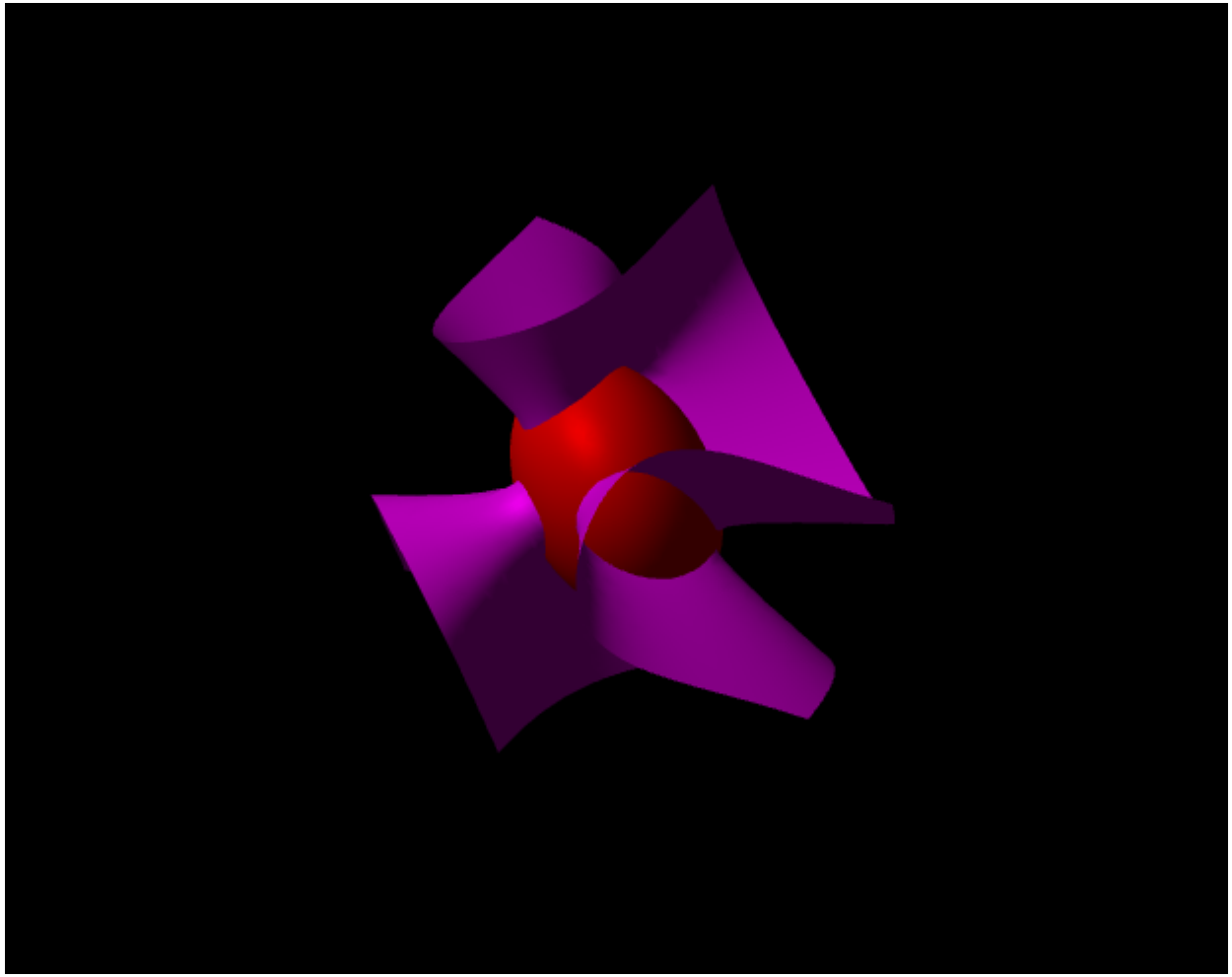

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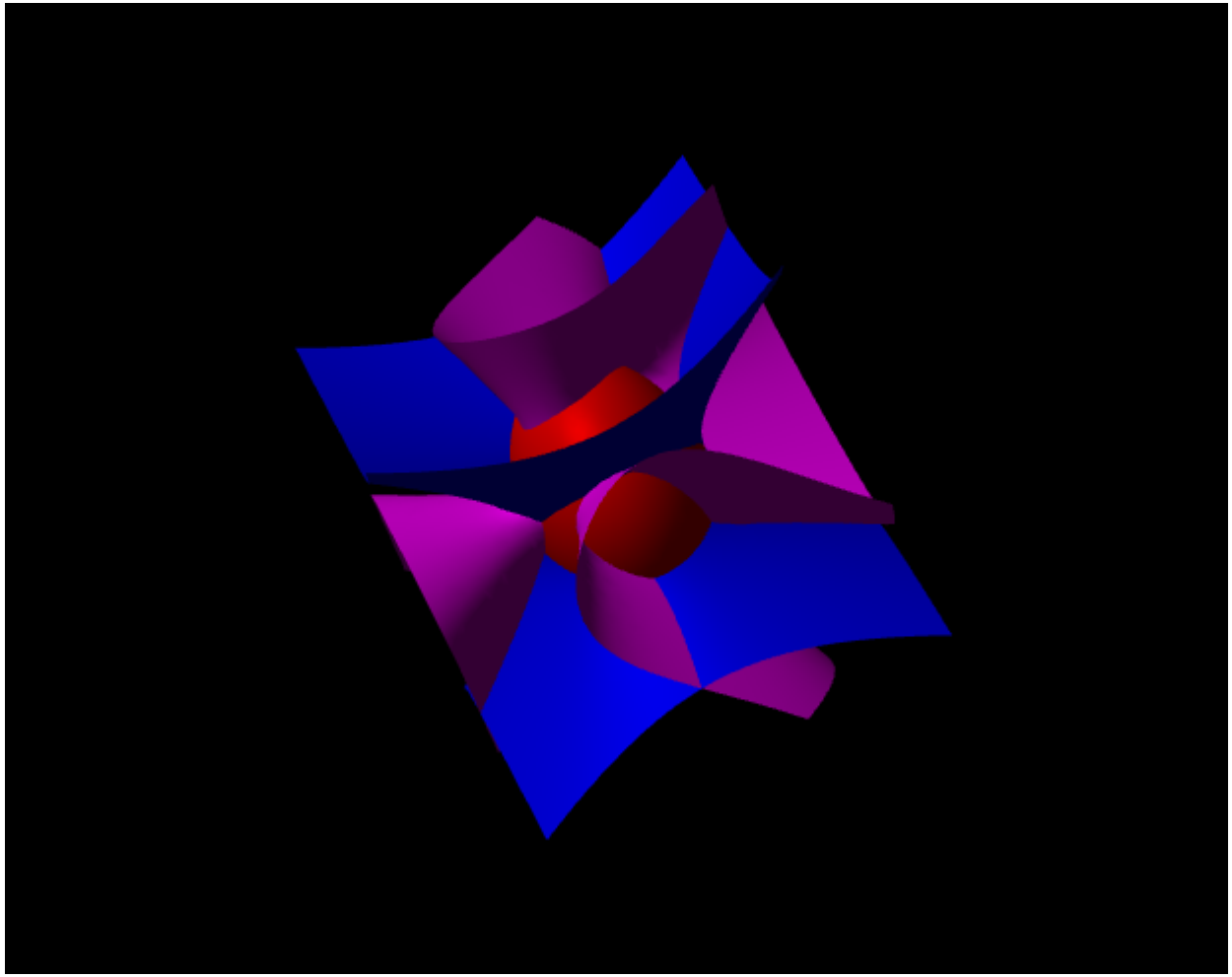

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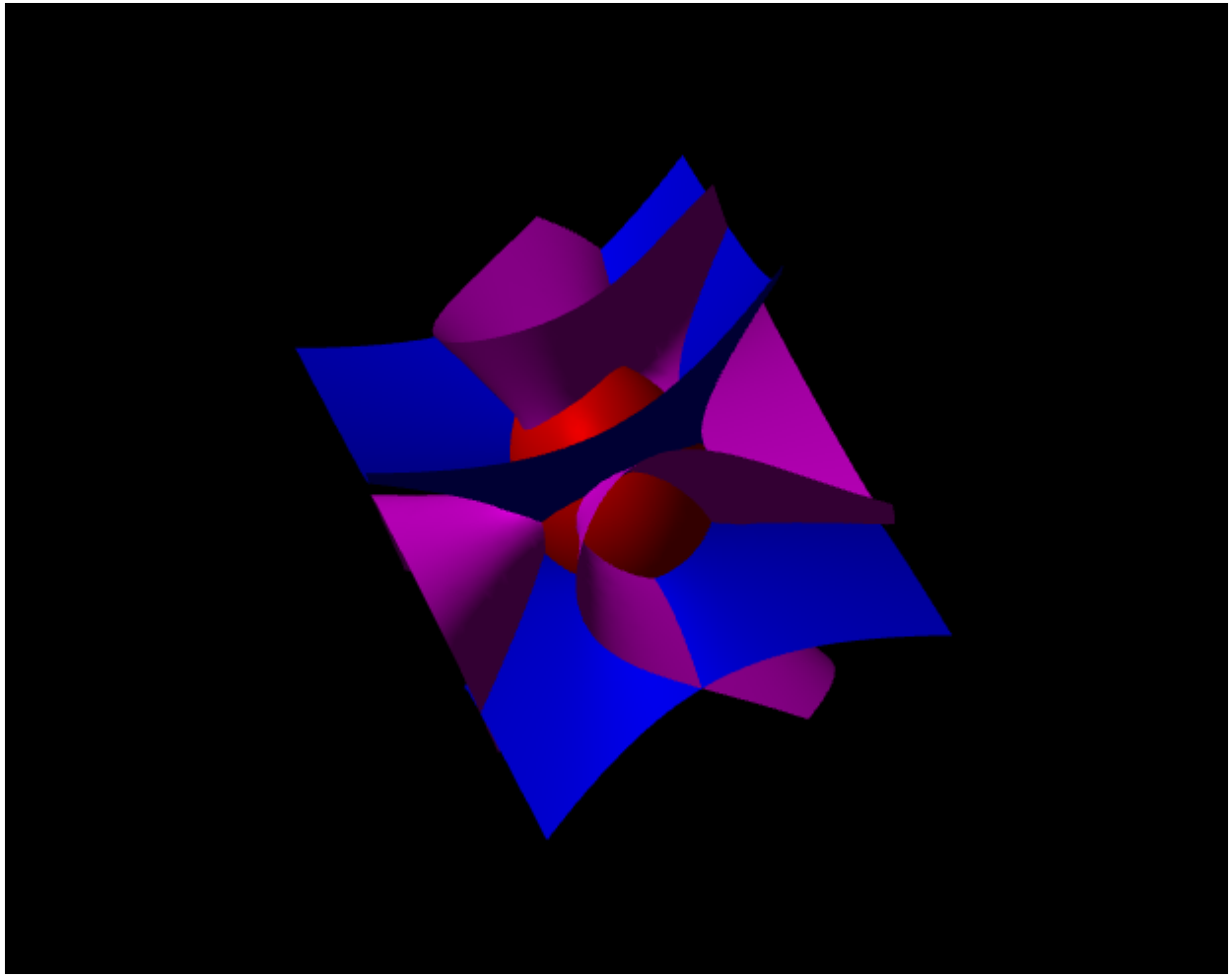

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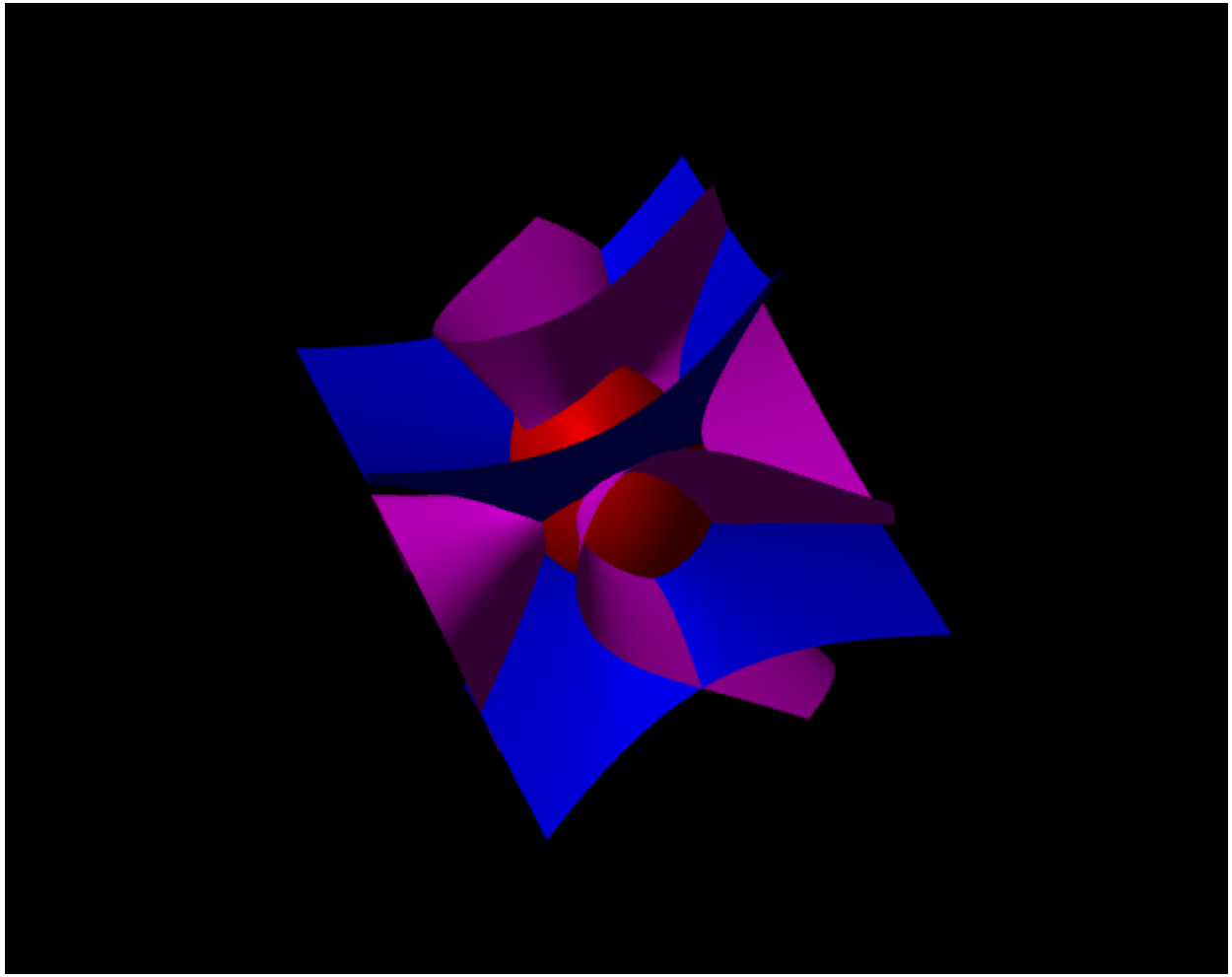

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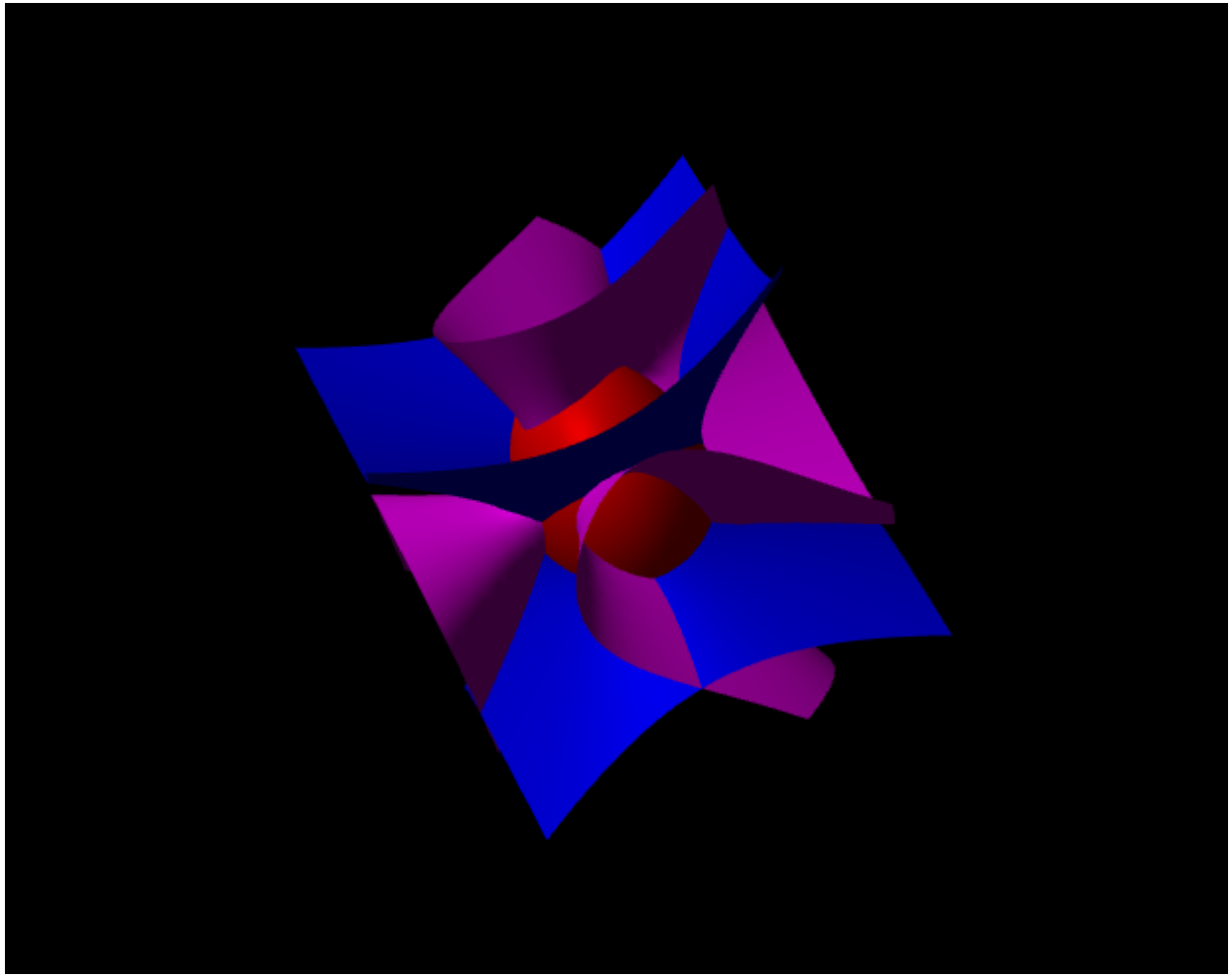

Then ${c_1}^2(X) = 12$, so ${c_1}^2(X)/3 = 4$. Hence $M = X \# 4\overline{\mathbb{CP}}_2$ does not admit Einstein metrics.

Theorem (L '01). *Let X be a minimal surface of general type, and let*

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In example:

$$\begin{aligned} c_1^2(X) &= 12 \\ k &= 4 \end{aligned}$$

So Theorem $\Longrightarrow$ *no* Einstein metric on $M = X \# 4\overline{\mathbb{CP}}_2$.

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End, Part IV