Cours de la chaire d'excellence

Fondation Sciences Mathématiques de Paris

Institut de Mathématiques
Jussieu - Paris Rive Gauche

Claude R. LeBrun

Stony Brook
Accueilli à l'IMJ-PRG (Sorbonne Univ., Univ. Paris Cité, CNRS)

Einstein Metrics, Four-Manifolds, and Differential Topology

Amphithéâtre Yvonne Choquet-Bruhat (bât. Perrin)

- Jeudi 19 mars 2026
- Jeudi 26 mars 2026
- Jeudi 2 avril 2026

Salle Pierre Grisvard (bât. Borel, 3e étage)

- Jeudi 9 avril 2026
- Jeudi 16 avril 2026
- Jeudi 23 avril 2026*

de 14h à 17h15

Institut Henri Poincaré

11, rue Pierre et Marie Curie, 75005 Paris

*La durée totale du cours étant de 15h, une séance parmi les trois dernières indiquées sera supprimée, en concertation avec Claude LeBrun.

Informations et résumé du cours
www.sciencesmaths-paris.fr

Einstein Metrics,

Four-Manifolds, &

Differential Topology, III

Claude LeBrun
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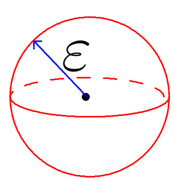

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Keep in mind $n = 4$ case:

$$g \longmapsto \int_M s_g^2 \, d\mu_g$$

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Critical $\Longrightarrow$ $\Delta(|s|^{\frac{n}{2}-2}s) = 0$ as distribution.

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$\therefore$ any critical metric satisfies $|s|^{\frac{n}{2}-2}s = \text{constant}.$

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So:

Critical points are Einstein or scalar-flat ($s \equiv 0$).

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Try to find Einstein metrics by minimizing?

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Interesting Integral Infimum Invariant. . .

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Theorem. *Let M be a compact simply connected n-manifold,*

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Invariance under surgery in codimension ≥ 3 is key.

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Builds on earlier work by Stolz concerning spin case.

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Theorem. *Let M be a compact simply connected n-manifold, $n \geq 3$. If $n \neq 4$, $\mathcal{I}_s(M) = 0$.*

One key ingredient:

If smooth compact M^n, $n \neq 2$, admits a metric g with $s > 0$, then also admits a metric $\tilde{g}$ with $s \equiv 0$.

Perelman: S^3 is only cpt simply-connected M^3.

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is realized by an Einstein metric g_j with $\lambda < 0$.

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When this happens, we then have

$$\mathcal{I}_s(M) = 32\pi^2(2\chi + 3\tau)(M).$$

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The proof is based on Seiberg-Witten theory, and only works in dimension 4.

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In higher dimensions, Kähler-Einstein metrics with $\lambda < 0$ also exist in profusion, but they do not enjoy this privileged position among Riemannian metrics!

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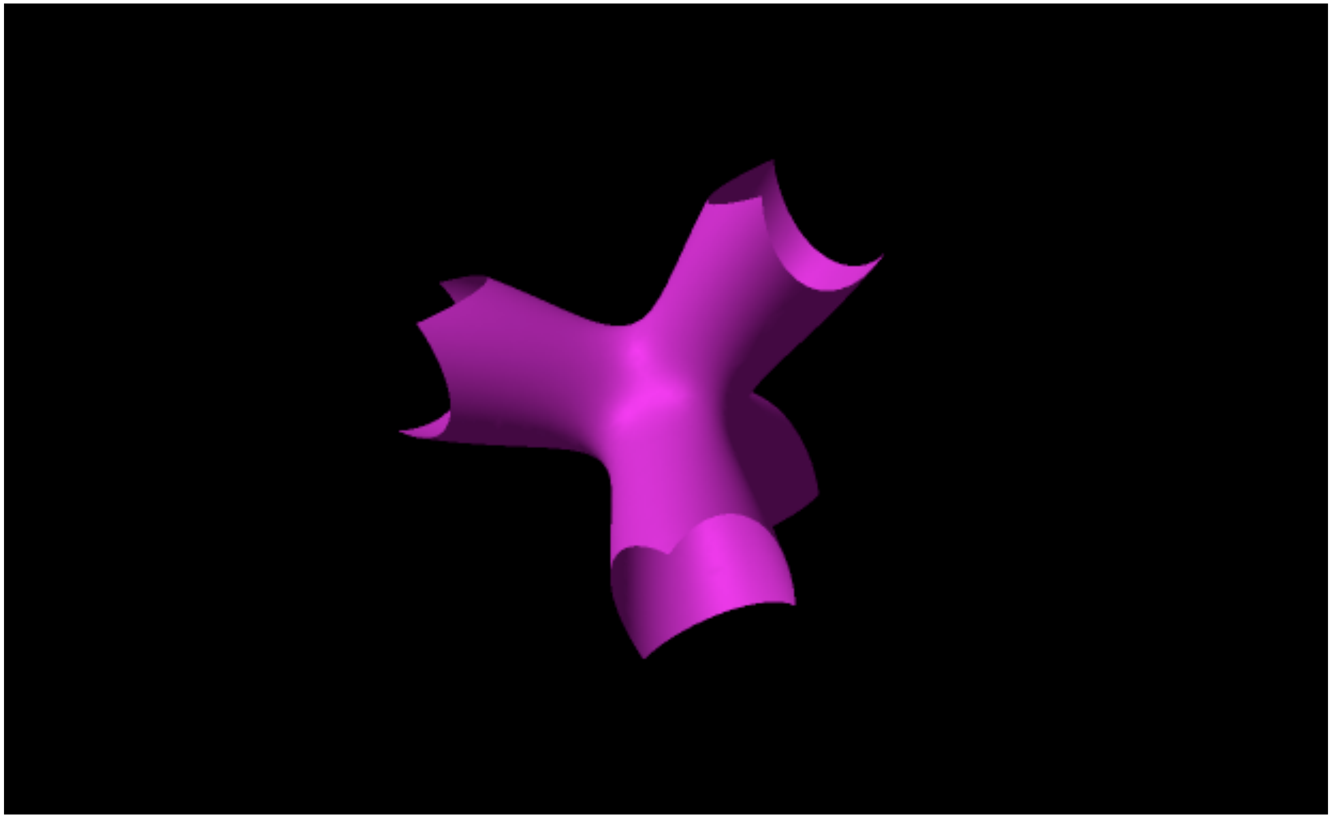

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If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

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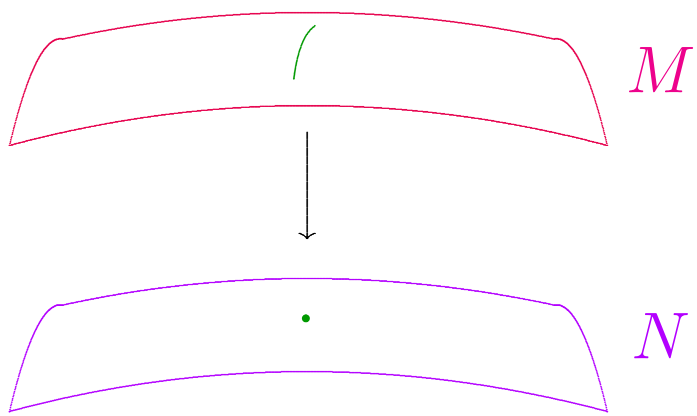
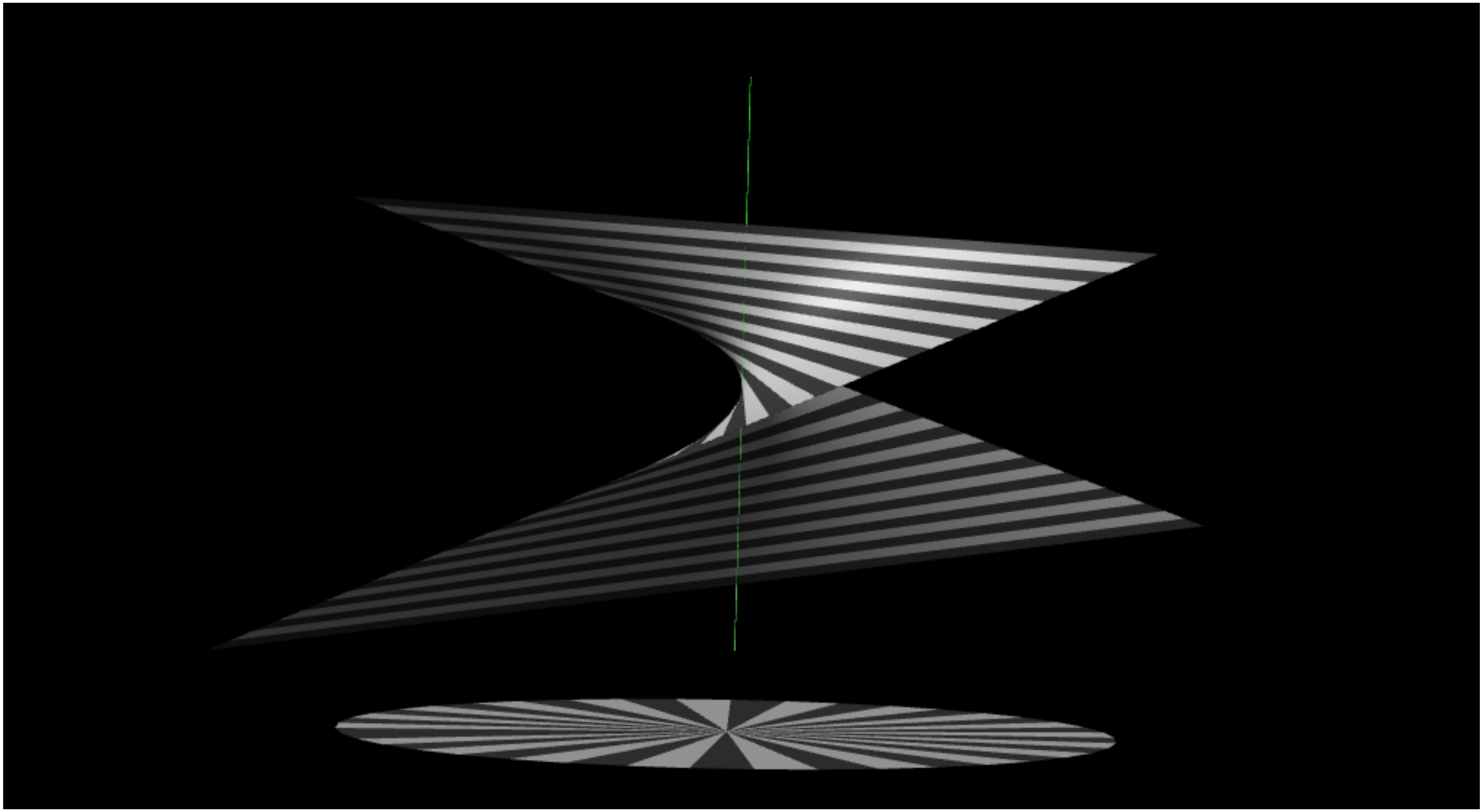
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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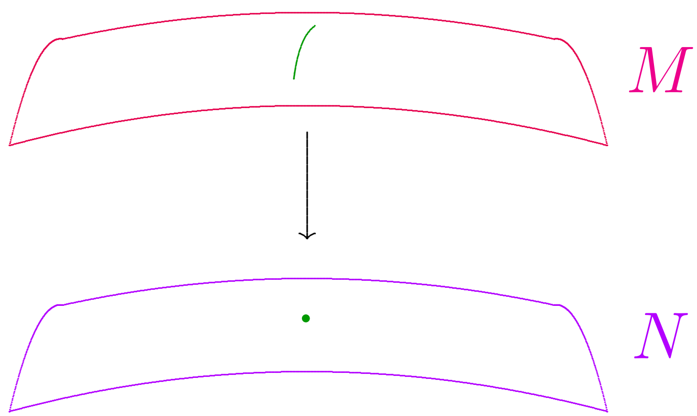

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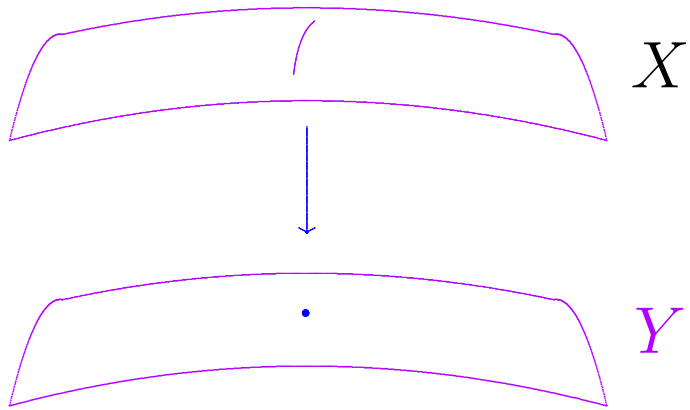

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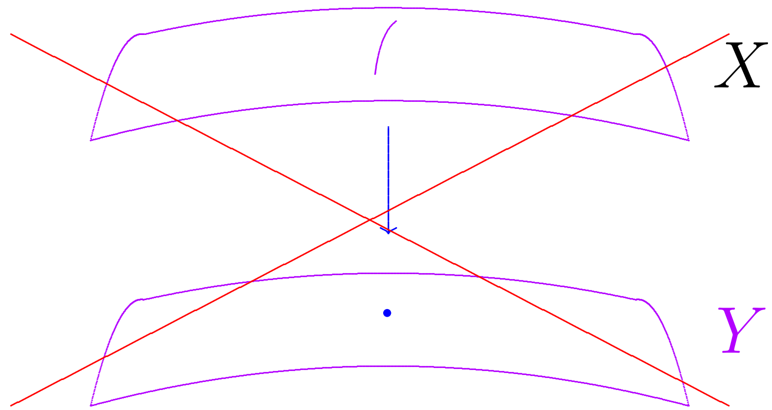

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is the "canonical line bundle" of M.

Kodaira Classification

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Notice that $c_1^2 > 0 \Longrightarrow \mathrm{Kod}(X) \in \{-\infty, 2\}$, and that X must be of Kähler type.

Grauert: happens because

$$\begin{aligned}
\chi(X, \mathcal{O}(K^{\otimes \ell})) &= h^0(\mathcal{O}(K^{\otimes \ell})) - h^1(\mathcal{O}(K^{\otimes \ell})) + h^2(\mathcal{O}(K^{\otimes \ell})) \\
&= h^0(\mathcal{O}(K^{\otimes \ell})) - h^1(\mathcal{O}(K^{\otimes \ell})) + h^0(\mathcal{O}(K^{\otimes (1-\ell)})) \\
&= \ell(\ell-1)\frac{c_1^2}{2} + \chi(X, \mathcal{O})
\end{aligned}$$

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$$c_1^2(M) = c_1^2(X \# k\overline{\mathbb{CP}}_2) = c_1^2(X) - k.$$

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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Theorem (Hitchin-Thorpe Inequality)**.** *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler. The latter case happens only if M finitely covered by flat T^4 or $K3$.

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Proposition. *If (M, J) compact complex surface, and if M admits Einstein metric g (unrelated to J) with $\lambda \neq 0$, then*

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and M admits a symplectic structure.

Intermission

Dirac Operators and Scalar Curvature

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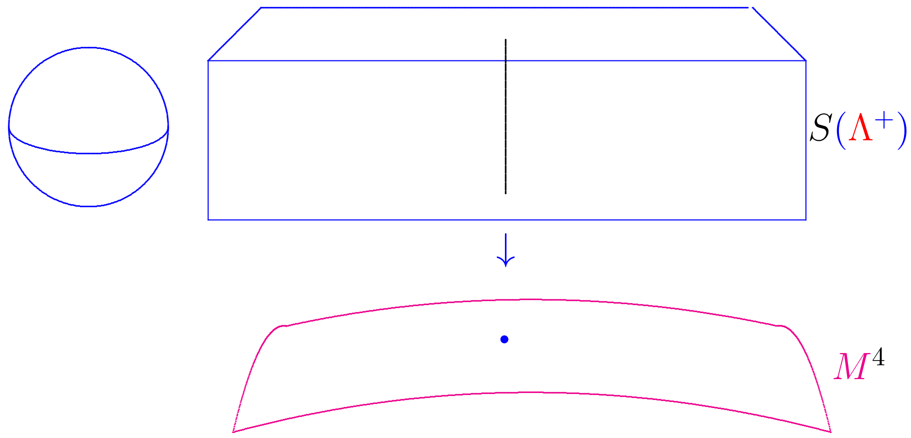

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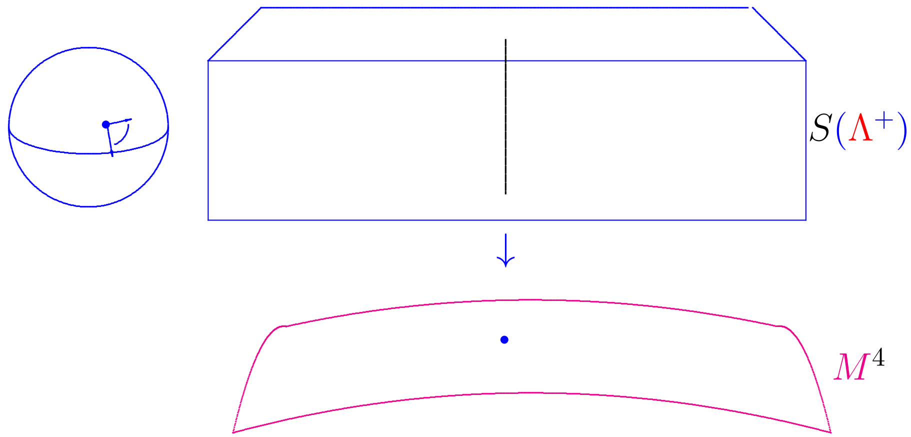

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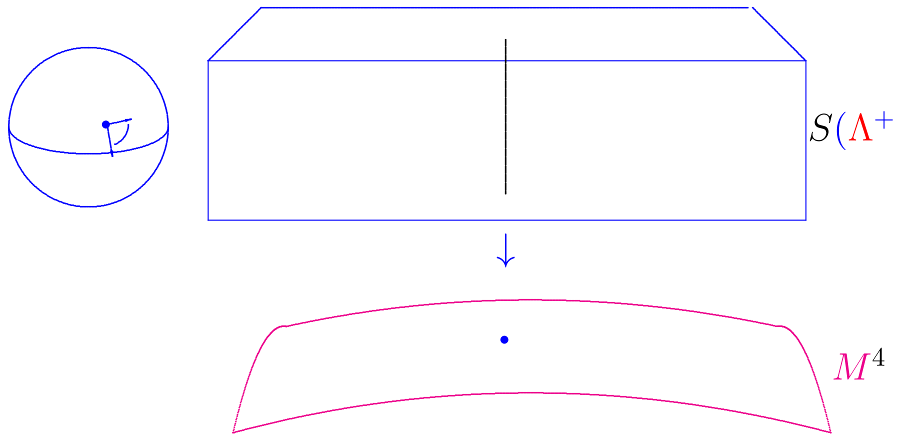

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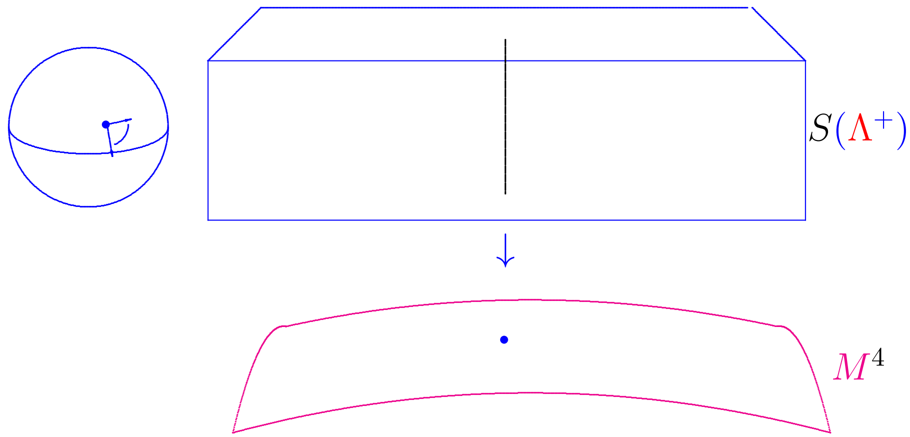

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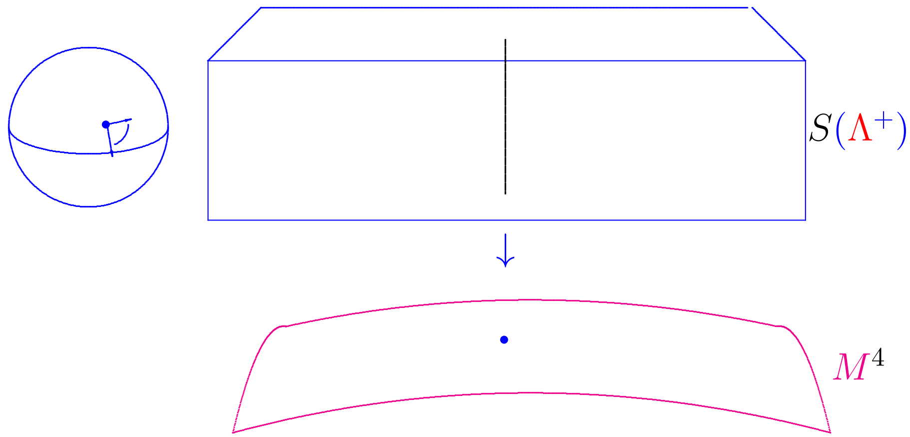

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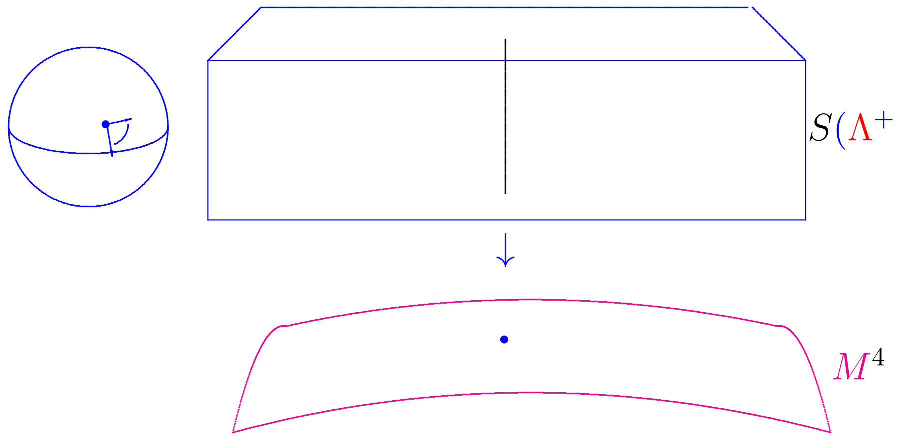

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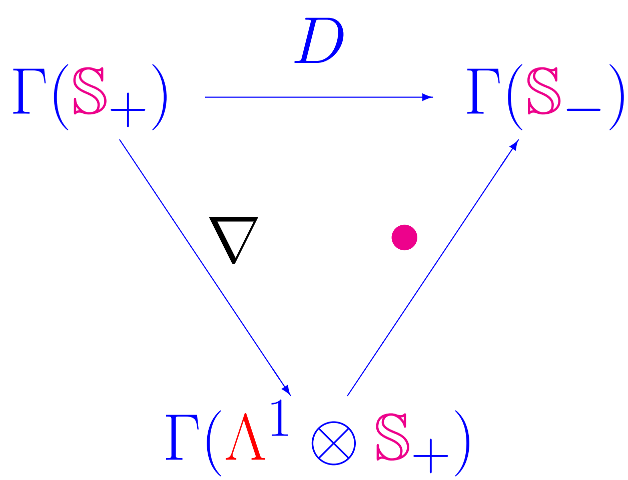

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Example. $\nexists$ metric of $s > 0$ on $K3$.

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Need a new technology to handle these cases!

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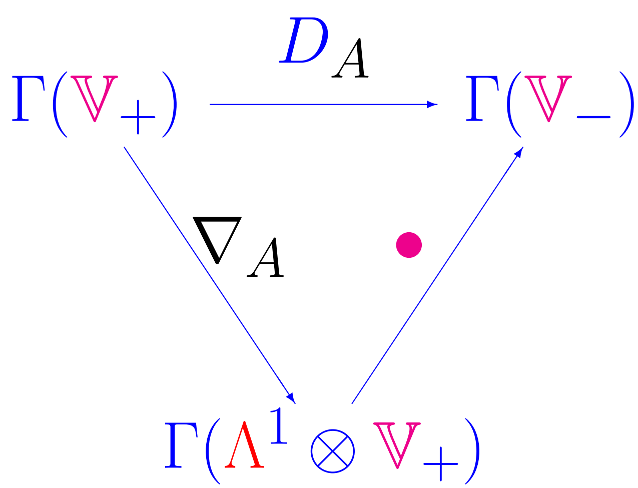

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Non-linear, but elliptic once 'gauge-fixing'

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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Compactness: Implies C^0 bound on Φ:

At maximum of Φ, $\Delta|\Phi|^2 \geq 0$, so

$$0 \geq s|\Phi|^2 + |\Phi|^4$$

and hence $|\Phi|^2 \leq -s$, unless $\Phi \equiv 0$. Hence

$$|\Phi| \leq \sqrt{\max |s_-|}$$

everywhere!

Bootstrapping with gauge-fixed equations, one gets L^p_k bounds for (Φ, A) for all k, p.

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SW invariant $\in \mathbb{Z}_2$ means mod-2 mapping degree.

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Implies non-existence of metrics g for which $s > 0$.

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler. The latter case happens only if M finitely covered by flat T^4 or $K3$.

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Key point: SW $\Rightarrow s > 0$ impossible when Kod $= 2$.

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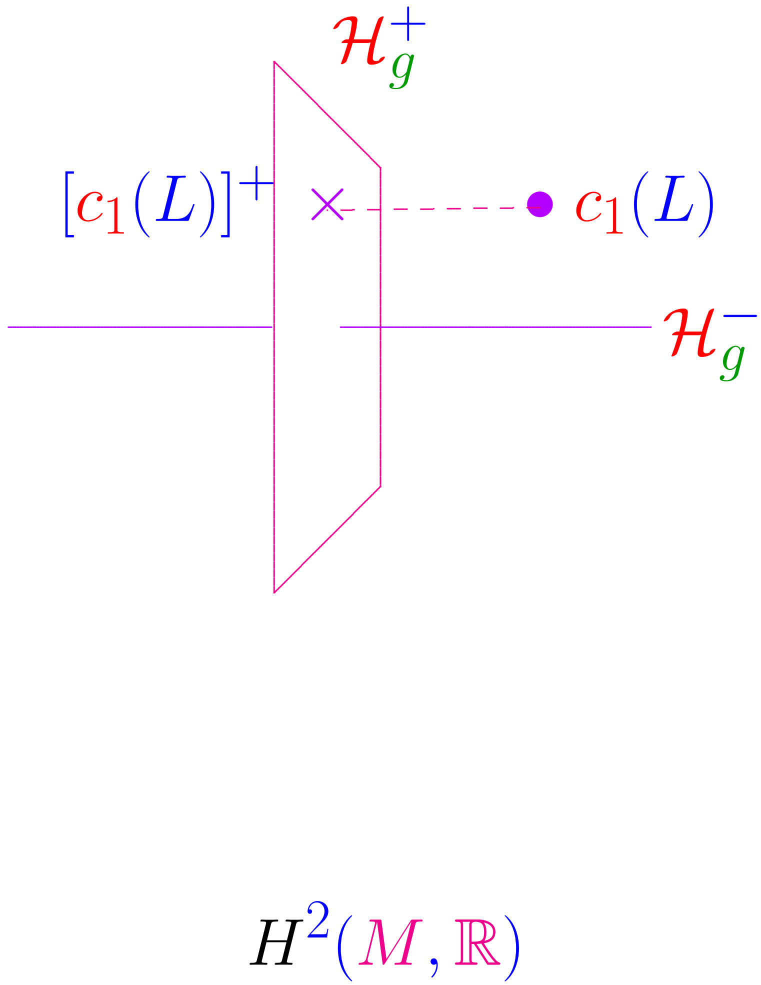
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by Weitzenböck for $(d+d^*)^2$. Hence

$$0 \geq \int \left[\left(s - \sqrt{6}|W_+|\right) |\psi|^2 + \frac{3}{2} f |\psi|^3\right] d\mu$$

Hölder inequality $\Longrightarrow$

$$\left(\int f^4 d\mu\right)^{1/3} \left(\int \left|s - \sqrt{6}|W_+|\right|^3 f^{-2} d\mu\right)^{2/3} \geq \frac{9}{4} \int f^2 |\psi|^2 d\mu$$

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End of Lecture III