Cours de la chaire d'excellence

Fondation Sciences Mathématiques de Paris

Institut de Mathématiques
Jussieu - Paris Rive Gauche

Claude R. LeBrun

Stony Brook
Accueilli à l'IMJ-PRG (Sorbonne Univ., Univ. Paris Cité, CNRS)

Einstein Metrics, Four-Manifolds, and Differential Topology

Amphithéâtre Yvonne Choquet-Bruhat (bât. Perrin)

- Jeudi 19 mars 2026
- Jeudi 26 mars 2026
- Jeudi 2 avril 2026

Salle Pierre Grisvard (bât. Borel, 3e étage)

- Jeudi 9 avril 2026
- Jeudi 16 avril 2026
- Jeudi 23 avril 2026*

de 14h à 17h15

Institut Henri Poincaré
11, rue Pierre et Marie Curie, 75005 Paris

*La durée totale du cours étant de 15h, une séance parmi les trois dernières indiquées sera supprimée, en concertation avec Claude LeBrun.

Informations et résumé du cours
WWW.SCIENCESMATHS-PARIS.FR

Einstein Metrics,

Four-Manifolds, &

Differential Topology, II

Claude LeBrun
Stony Brook University

Cours de la Chaire d'Excellence
Fondation Sciences Mathématiques de Paris
Institut Henri Poincaré, jeudi 26 mars 2026

Intersection Form, Revisited:

Intersection Form, Revisited:

Assume M^4 smooth, compact, connected.

Intersection Form, Revisited:

Intersection Form, Revisited:

Version over $\mathbb{R}$: (deRham cohomology version)

$$\bullet : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$([\varphi] , [\psi]) \longmapsto \int_M \varphi \wedge \psi$$

Intersection Form, Revisited:

Version over $\mathbb{R}$:

$$
\begin{aligned}
\bullet : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \quad \mathbb{R} \\
(\ [\varphi]\ ,\ [\psi]\ ) \qquad &\longmapsto \int_M \varphi \wedge \psi
\end{aligned}
$$

Diagonalize:

$$
\left[\begin{array}{cccccc}
\underbrace{\begin{array}{ccc} +1 & & \\ & \ddots & \\ & & +1 \end{array}}_{b_+(M)} & \\
& b_-(M)\left\{\begin{array}{ccc} -1 & & \\ & \ddots & \\ & & -1 \end{array}\right.
\end{array}\right].
$$

Intersection Form, Revisited:

Version over $\mathbb{R}$:

$$\bullet : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(\,[\varphi]\,,\,[\psi]\,) \longmapsto \int_M \varphi \wedge \psi$$

Signature

$$\tau(M) = b_+(M) - b_-(M)$$

Intersection Form, Revisited:

Version over $\mathbb{R}$:

$$\bullet : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$([\varphi], [\psi]) \longmapsto \int_M \varphi \wedge \psi$$

Signature

$$\tau(M) = b_+(M) - b_-(M)$$

Thom-Hirzebruch signature formula:

$$\tau(M) = \frac{1}{3} p_1(M)$$

Intersection Form, Revisited:

Version over $\mathbb{R}$:

$$\bullet : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(\ [\varphi]\ ,\ [\psi]\ ) \longmapsto \int_M \varphi \wedge \psi$$

Signature

$$\tau(M) = b_+(M) - b_-(M)$$

Thom-Hirzebruch signature formula:

$$\tau(M) = \frac{1}{3} p_1(M) = \frac{1}{3} \int_M \wp_1(TM)$$

Intersection Form, Revisited:

Version over $\mathbb{R}$:

$$\bullet : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$([\varphi] , [\psi]) \longmapsto \int_M \varphi \wedge \psi$$

Intersection Form, Revisited:

Version over $\mathbb{R}$: (Singular cohomology version)

$$\bullet : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Intersection Form, Revisited:

Version over $\mathbb{R}$: (Singular cohomology version)

$$
\begin{aligned}
\bullet : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \quad \mathbb{R} \\
(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \qquad &\longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle
\end{aligned}
$$

Oriented homotopy invariant!

Intersection Form, Revisited:

Version over $\mathbb{R}$: (Singular cohomology version)

$$\bullet : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$(\,\mathbf{a}\,,\ \mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Oriented homotopy invariant!

So $\tau(M)$ is, too!

Intersection Form, Revisited:

Version over $\mathbb{R}$: (Singular cohomology version)

$$\bullet : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(\, \mathbf{a}\, ,\ \mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\mathbf{a}, \mathbf{b}) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Vanishes on torsion!

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Vanishes on torsion!

$$H^2(M, \mathbb{Z})/\text{torsion} \quad \hookrightarrow \quad H^2(M, \mathbb{R})$$

as lattice.

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M]\rangle$$

Vanishes on torsion!

$$H^2(M,\mathbb{Z})/\text{torsion} \quad \hookrightarrow \quad H^2(M,\mathbb{R})$$

Integral lattice: $[\varphi] \in H^2_{dR}(M,\mathbb{R})$ such that

$$\int_\Sigma \varphi \in \mathbb{Z} \qquad \forall \Sigma^2 \subset M \text{ compact oriented.}$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\,\mathbf{a}\,,\,\mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

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Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Geometric interpretation:

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Geometric interpretation:

$$H^2(M,\mathbb{Z}) = \{C^\infty \text{complex line bundles } L \to M\}/\text{isomorphism}$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\,\mathbf{a}\,,\,\mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Geometric interpretation:

$$H^2(M,\mathbb{Z}) \xleftarrow{c_1} \{C^\infty \text{complex line bundles } L \to M\}/\text{isomorphism}$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M]\rangle$$

Geometric interpretation: (Čech cohomology)

$$H^2(M,\mathbb{Z}) \stackrel{c_1}{\longleftarrow} \{C^\infty \text{complex line bundles } L \to M\}/\text{isomorphism}$$

$$0 \to \mathbb{Z} \stackrel{2\pi i \cdot}{\to} \mathscr{E} \stackrel{\exp}{\to} \mathscr{E}^\times \to 0$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Geometric interpretation: (Čech cohomology)

$$H^2(M,\mathbb{Z}) \xleftarrow{c_1} \{C^\infty \text{complex line bundles } L \to M\}/\text{isomorphism}$$

$$\cdots H^1(M,\mathscr{E}) \to H^1(M,\mathscr{E}^\times) \to H^2(M,\mathbb{Z}) \to H^2(M,\mathscr{E}) \cdots$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\begin{array}{rcl} \bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (\ \mathbf{a}\ ,\ \mathbf{b}\ ) & \longmapsto & \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle \end{array}$$

Geometric interpretation: (Čech cohomology)

$$H^2(M,\mathbb{Z}) \overset{c_1}{\longleftarrow} \{C^\infty \text{complex line bundles } L \to M\}/\text{isomorphism}$$

$$0 \to H^1(M, \mathscr{E}^\times) \to H^2(M,\mathbb{Z}) \to 0$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Geometric interpretation:

$$H^2(M, \mathbb{Z}) \stackrel{c_1}{\longleftarrow} \{C^\infty \text{complex line bundles } L \to M\}/\text{isomorphism}$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
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Geometric interpretation:

$$H^2(M,\mathbb{Z}) = \{C^\infty \text{complex line bundles } L \to M\}/\text{isomorphism}$$

Intersection Form, Revisited:

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$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Geometric interpretation:

$$H^2(M, \mathbb{Z}) = H_2(M, \mathbb{Z})$$

via Poincaré duality.

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\begin{array}{rcl} \bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (\,\mathbf{a}\,,\,\mathbf{b}\,) & \longmapsto & \langle \mathbf{a}\cup\mathbf{b}, [M]\rangle \end{array}$$

Geometric interpretation:

$$H^2(M,\mathbb{Z}) = H_2(M,\mathbb{Z})$$

via Poincaré duality.

Line bundle L $\longmapsto$ zero set Σ generic section

Intersection Form, Revisited:

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Geometric interpretation:

$$H^2(M,\mathbb{Z}) = H_2(M,\mathbb{Z})$$

via Poincaré duality.

Line bundle $L \quad \longmapsto \quad$ zero set Σ generic section

Pairing: $c_1(L_1) \bullet c_1(L_2)$

Intersection Form, Revisited:

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$$\begin{array}{rcl} \bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (\,\mathbf{a}\,,\,\mathbf{b}\,) & \longmapsto & \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle \end{array}$$

Geometric interpretation:

$$H^2(M,\mathbb{Z}) = H_2(M,\mathbb{Z})$$

via Poincaré duality.

Line bundle L $\longmapsto$ zero set Σ generic section

Pairing: $c_1(L_1) \bullet c_1(L_2) = \langle c_1(L_1), [\Sigma_2] \rangle$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\,\mathbf{a}\,,\,\mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Geometric interpretation:

$$H^2(M,\mathbb{Z}) = H_2(M,\mathbb{Z})$$

via Poincaré duality.

Line bundle L $\longmapsto$ zero set Σ generic section

Pairing: $c_1(L_1) \bullet c_1(L_2) = \langle c_1(L_1), [\Sigma_2] \rangle$
$= \#$ zeroes generic section $L_1|_{\Sigma_2}$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
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Geometric interpretation:

$$H^2(M, \mathbb{Z}) = H_2(M, \mathbb{Z})$$

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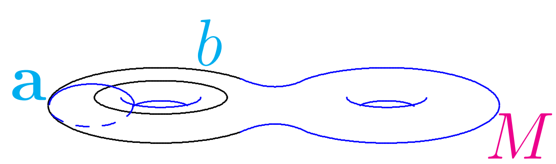
via Poincaré duality.



Intersection Form, Revisited:

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Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Special case: self-intersection

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

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Special case: self-intersection

$$\mathbf{a} \longmapsto \mathbf{a} \bullet \mathbf{a} \in \mathbb{Z}$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

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Special case: self-intersection

$$\mathbf{a} \quad \longmapsto \quad \mathbf{a} \bullet \mathbf{a} \quad \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} \quad = \quad \#(\Sigma \cap \widetilde{\Sigma})$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Special case: self-intersection

$$\mathbf{a} \quad \longmapsto \quad \mathbf{a} \bullet \mathbf{a} \quad \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} \quad = \quad \#(\Sigma \cap \widetilde{\Sigma})$$

counted with signs.

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Special case: self-intersection

$$\mathbf{a} \quad \longmapsto \quad \mathbf{a} \bullet \mathbf{a} \quad \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} \quad = \int_{\Sigma} c_1(L)$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\,\mathbf{a}\,,\;\mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Special case: self-intersection

$$\mathbf{a} \longmapsto \mathbf{a} \bullet \mathbf{a} \quad \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} \quad = \int_{\Sigma} c_1(L)$$

where $L|_{\Sigma} \to \Sigma$ is normal bundle.

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\begin{array}{rcl} \bullet_{\mathbb{Z}} : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (\,\mathbf{a}\,,\ \mathbf{b}\,) & \longmapsto & \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle \end{array}$$

Special case: self-intersection

$$\mathbf{a} \quad \longmapsto \quad \mathbf{a} \bullet \mathbf{a} \quad \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} \quad = \int_{\Sigma} c_1(\nu)$$

where $\nu \to \Sigma$ is normal bundle $\nu = (T\Sigma)^{\perp}$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\mathbf{a}, \mathbf{b}) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Special case: self-intersection

$$\mathbf{a} \longmapsto \mathbf{a} \bullet \mathbf{a} \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} = \int_{\Sigma} \mathsf{e}(\nu)$$

where $\nu \to \Sigma$ is normal bundle $\nu = (T\Sigma)^{\perp}$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\, \mathbf{a} \, , \, \mathbf{b} \,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

But

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\varrho} \mathbb{Z}_2 \to 0$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\,\mathbf{a}\,,\,\mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

But

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\varrho} \mathbb{Z}_2 \to 0$$

induces

$$\varrho : H^2(\Sigma, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}_2)$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
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But

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\varrho} \mathbb{Z}_2 \to 0$$

induces

$$\varrho : H^2(\Sigma, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}_2)$$

and

$$\varrho[\mathsf{e}(\nu)] = w_2(\nu).$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\ \mathbf{a}\ ,\ \mathbf{b}\ ) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle$$

Special case: self-intersection

$$\mathbf{a} \quad \longmapsto \quad \mathbf{a} \bullet \mathbf{a} \quad \in \mathbb{Z}$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\begin{aligned} \bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ (\mathbf{a}, \mathbf{b}) &\longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M] \rangle \end{aligned}$$

Special case: self-intersection

$$\mathbf{a} \longmapsto \mathbf{a} \bullet \mathbf{a} \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

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Special case: self-intersection

$$\mathbf{a} \longmapsto \mathbf{a} \bullet \mathbf{a} \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} = \int_{\Sigma} \mathsf{e}(\nu)$$

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Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$\mathbf{a} \bullet \mathbf{a} = \int_{\Sigma} \mathsf{e}(\nu)$$

where $\nu \to \Sigma$ is normal bundle $\nu = (T\Sigma)^{\perp}$

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Special case: self-intersection

$$\mathbf{a} \quad \longmapsto \quad \mathbf{a} \bullet \mathbf{a} \quad \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$[\Sigma] \bullet [\Sigma] \quad \equiv \langle w_2(\nu), [\Sigma] \rangle \quad \mathrm{mod} 2$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\,\mathbf{a}\,,\ \mathbf{b}\,) \longmapsto \langle \mathbf{a} \cup \mathbf{b}, [M]\rangle$$

$$TM|_{\Sigma} = \nu \oplus T\Sigma$$

Intersection Form, Revisited:

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$$TM|_{\Sigma} = \nu \oplus T\Sigma$$

$$j^* w_2(TM) = w_2(\nu) + w_2(T\Sigma)$$

Intersection Form, Revisited:

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Special case: self-intersection

$$\mathbf{a} \longmapsto \mathbf{a} \bullet \mathbf{a} \in \mathbb{Z}$$

Poincaré dual Σ and generic small perturbation $\widetilde{\Sigma}$

$$[\Sigma] \bullet [\Sigma] \equiv \langle w_2(\nu), [\Sigma] \rangle \mod 2$$

where $\nu \to \Sigma$ is normal bundle $\nu = (T\Sigma)^{\perp}$

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$\therefore$ can detect w_2 by pairing with oriented surfaces!

Theorem (Wu). *Suppose that M^4 is compact, simply connected.*

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Otherwise, one says that the form is odd.

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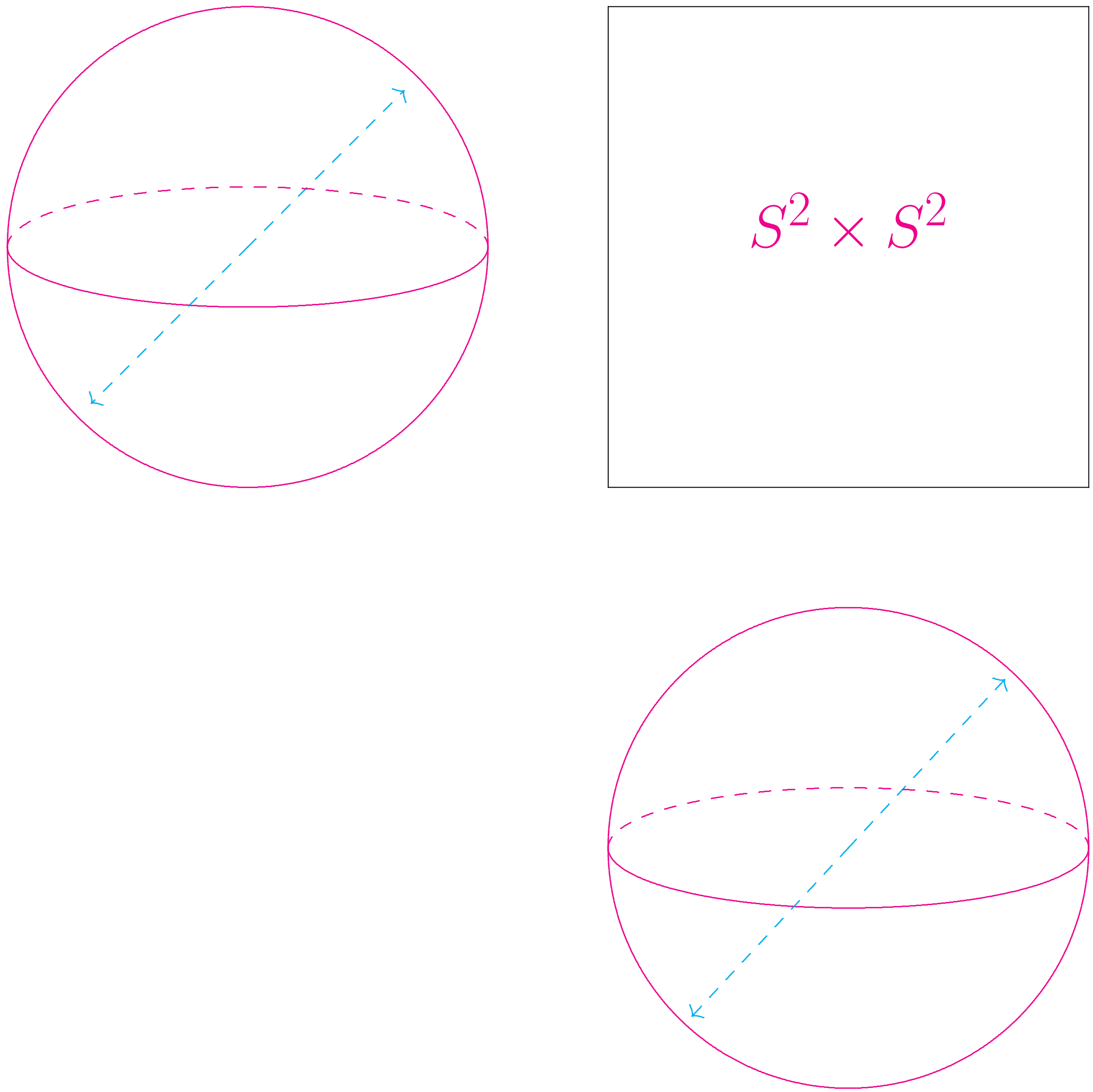
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To make Wu work in general, use intersection form

$$\bullet_{\mathbb{Z}_2} : H^2(M, \mathbb{Z}_2) \times H^2(M, \mathbb{Z}_2) \longrightarrow \quad \mathbb{Z}_2$$

with $\mathbb{Z}_2$ coefficients instead!

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Intersection Form, Revisited:

Version over $\mathbb{Z}$ via Poincaré duality:

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Can represent by $b_2 \times b_2$ integer matrix

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"Unimodular"

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"Intersection Matrix"

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Can represent by $b_2 \times b_2$ integer matrix of $\det = \pm 1$.

But any two similar matrices arise from same M^4.

Intersection Form, Revisited:

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Example.

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

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Example. $j\mathbb{CP}_2\#k\overline{\mathbb{CP}_2}$

Recall:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

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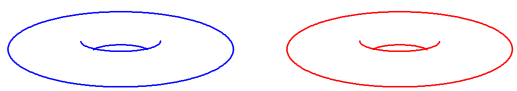
Connected sum #:



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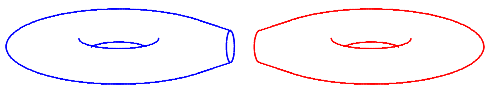

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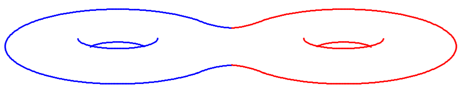

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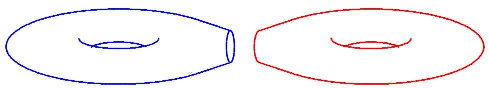

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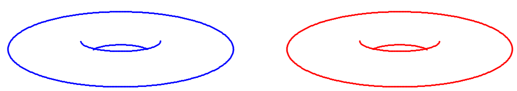

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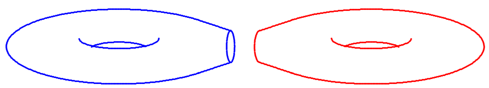

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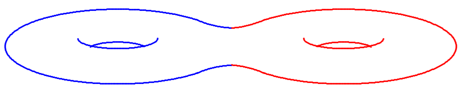

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$$\begin{bmatrix} \underbrace{\begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix}}_{j} & \\ & k\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{bmatrix}.$$

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$$\underbrace{\langle +1\rangle \oplus \cdots \oplus \langle +1\rangle}_{j} \oplus \underbrace{\langle -1\rangle \oplus \cdots \oplus \langle -1\rangle}_{k}$$

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"odd form"

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"odd form" (not even!)

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"odd form" (not even!)

$$w_2 \neq 0$$

Intersection Form, Revisited:

Version over $\mathbb{Z}$:

$$\bullet_{\mathbb{Z}} : H^2(M, \mathbb{Z})/\text{torsion} \times H^2(M, \mathbb{Z})/\text{torsion} \longrightarrow \mathbb{Z}$$

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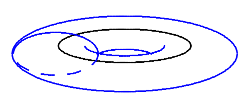

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$$(m, n) \bullet (m, n) = 2mn$$

"even form"

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$$\mathsf{H} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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$$\mathsf{H} \oplus \cdots \oplus \mathsf{H}$$

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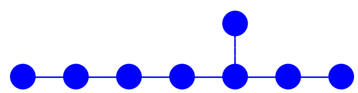
even, $w_2 = 0$, spin

Unimodular Forms:

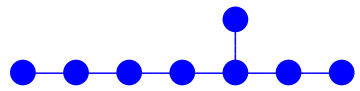
Example.



Unimodular Forms:

Example.

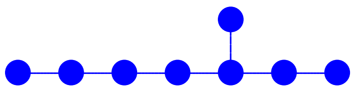
$$
\begin{bmatrix}
-2 & 1 & & & & & & \\
1 & -2 & 1 & & & & & \\
 & 1 & -2 & 1 & & & & \\
 & & 1 & -2 & 1 & & & \\
 & & & 1 & -2 & 1 & & 1 \\
 & & & & 1 & -2 & 1 & \\
 & & & & & 1 & -2 & \\
 & & & & 1 & & & -2
\end{bmatrix}
$$



Unimodular Forms:

Example.

$$\mathsf{E}_8 = \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & 1 \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & 1 & & & -2 \end{bmatrix}$$



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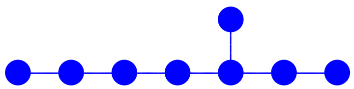
 & & & & 1 & -2 & 1 & \\
 & & & & & 1 & -2 & \\
 & & & & 1 & & & -2
\end{bmatrix}
$$



even, negative definite, signature -8.

Unimodular Forms:

Theorem. *Indefinite unimodular forms are completely classified by*

Unimodular Forms:

Theorem. *Indefinite unimodular forms are completely classified by rank,*

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Warning. Wildly false for definite forms!

Theorem (Milnor). *Two compact simply-connected 4-manifolds are orientedly homotopy equivalent*

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Atiyah-Singer:

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Atiyah-Singer:
new proof based on the index of the Dirac operator.

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$\Longrightarrow$ intersection form $\pm[j\mathsf{E}_8 \oplus k\mathsf{H}]$, with j even.

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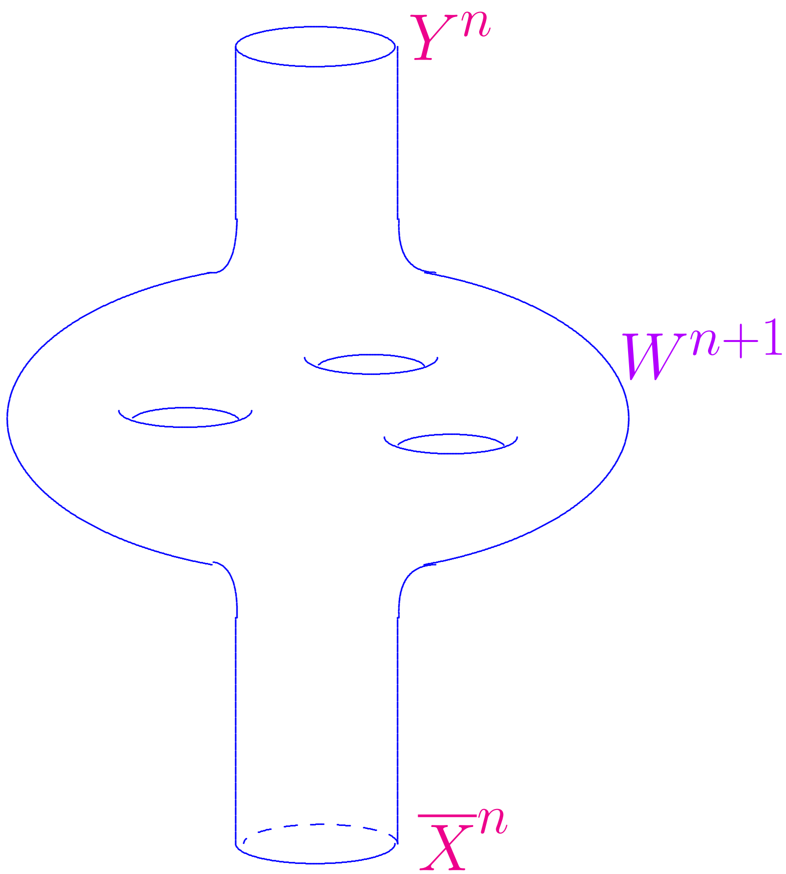
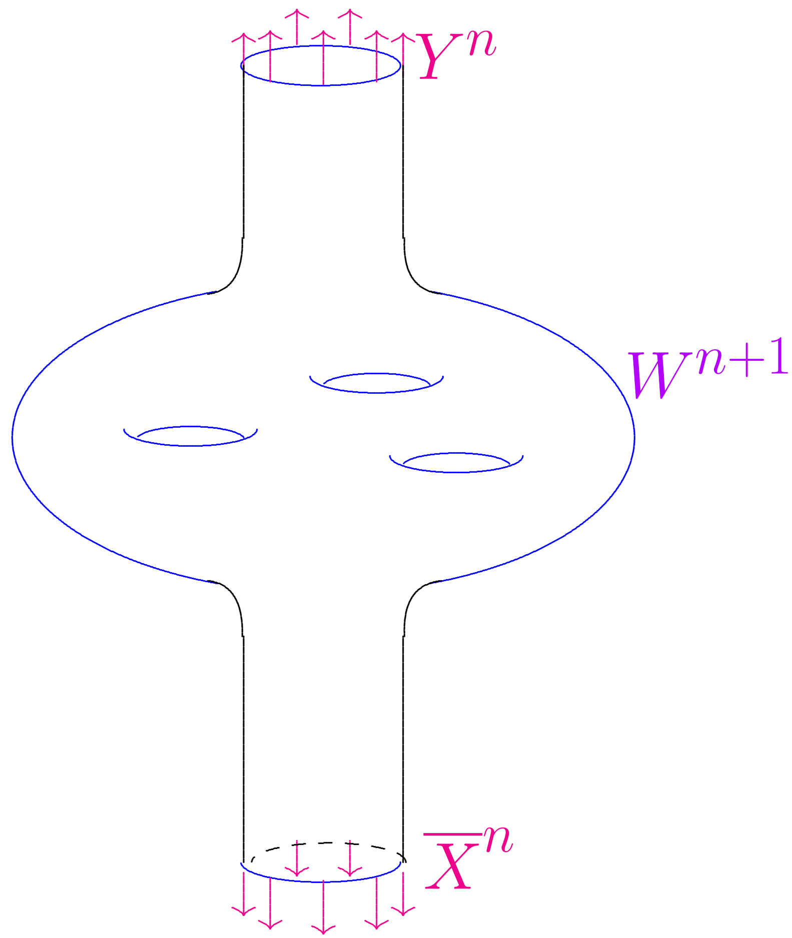
$$\langle 1 \rangle \oplus \langle 1 \rangle \oplus \cdots \oplus \langle 1 \rangle \oplus \langle 1 \rangle$$

.



Cobordism



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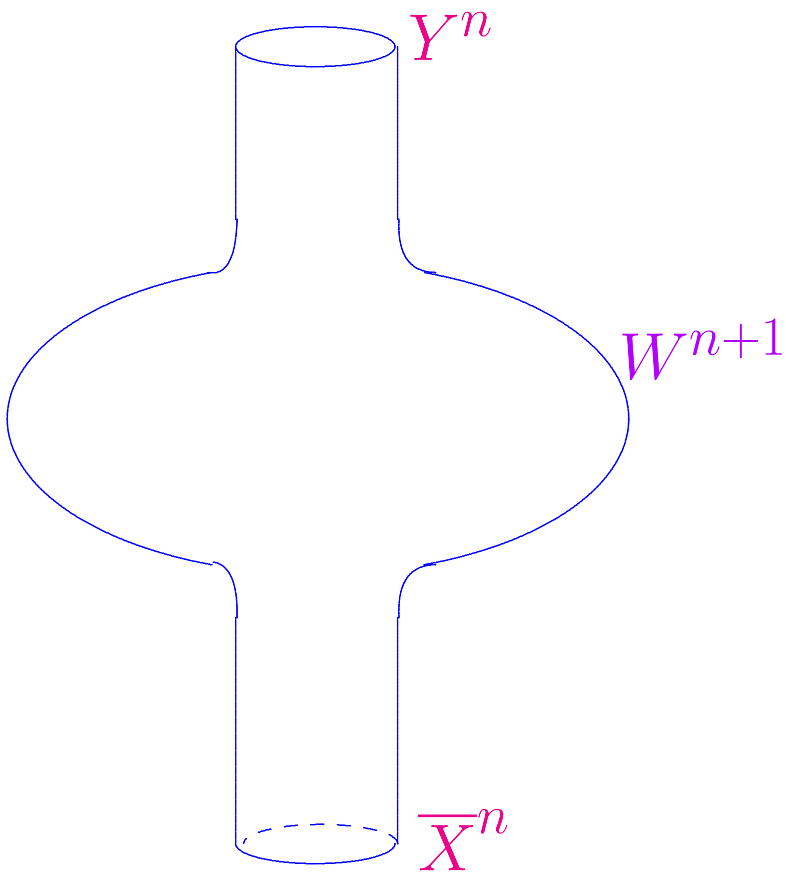
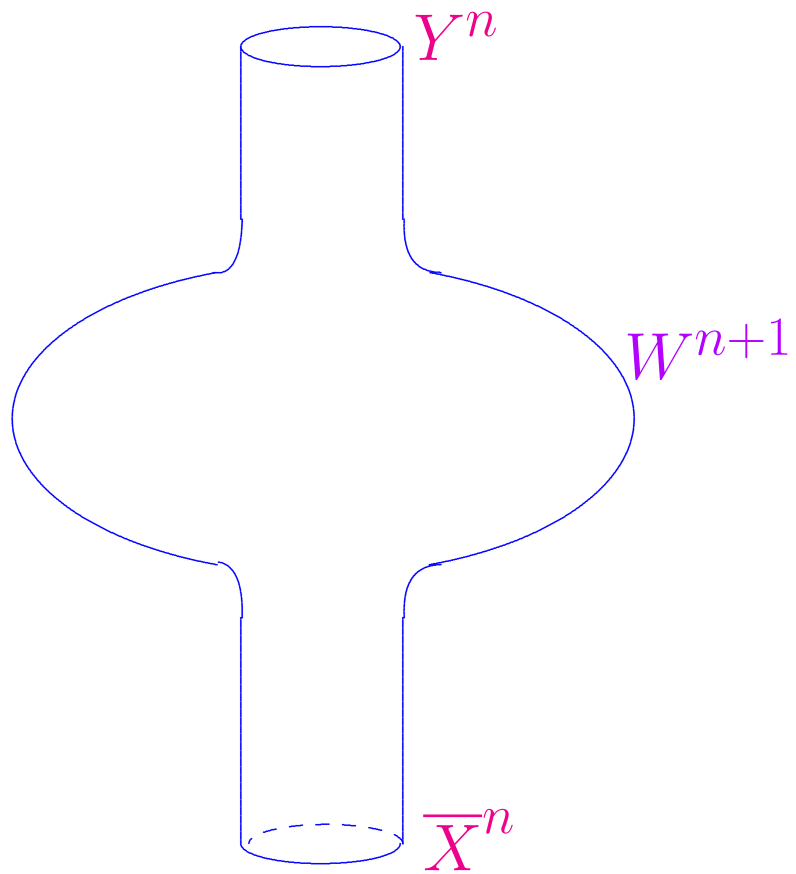
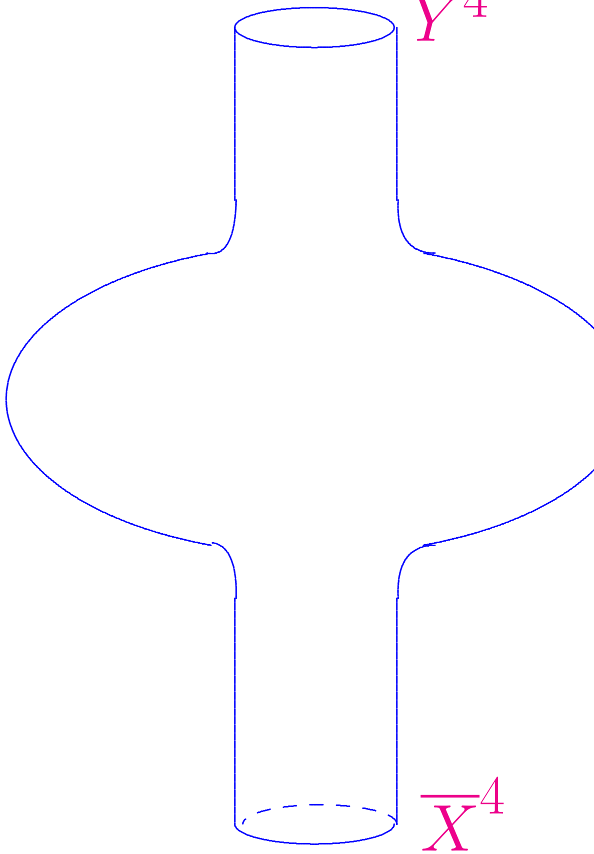


h-Cobordism

if $X \hookrightarrow W$, $Y \hookrightarrow W$ both homotopy equivalences



Smale: Suppose that X^n is h-cobordant to Y^n. If $\pi_1 = 0$ and $n > 4$, then X is diffeomorphic to Y.



Wall: Suppose that X^4 homotopy equivalent to Y^4.
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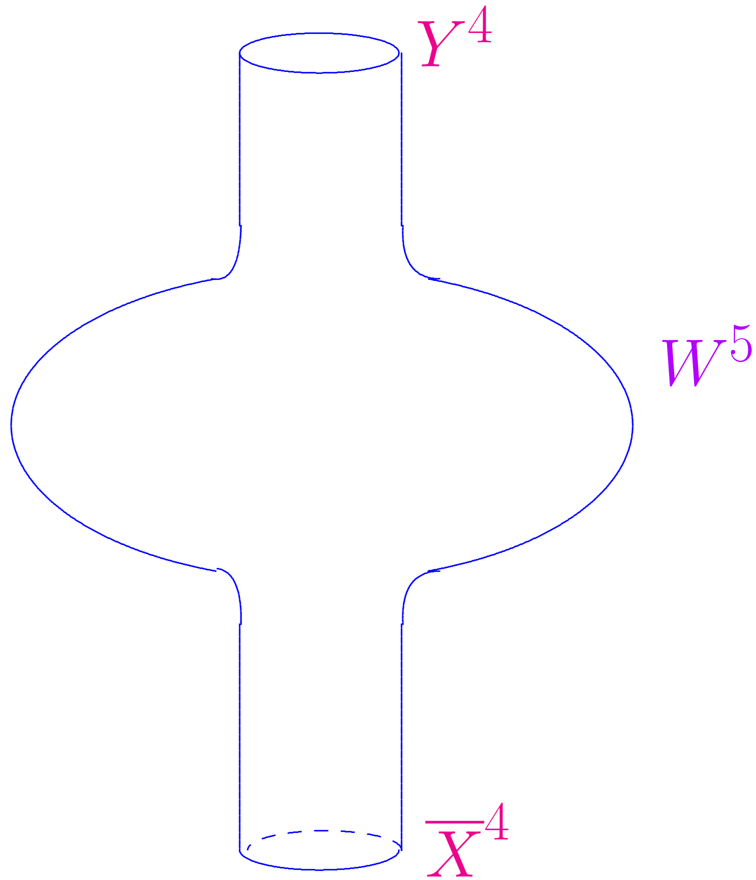

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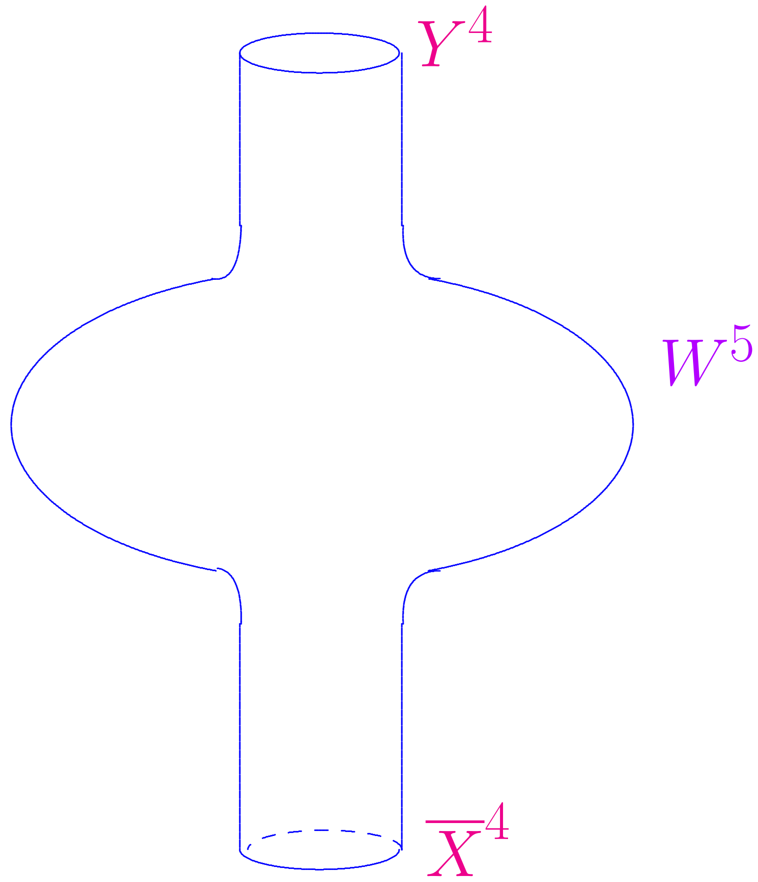
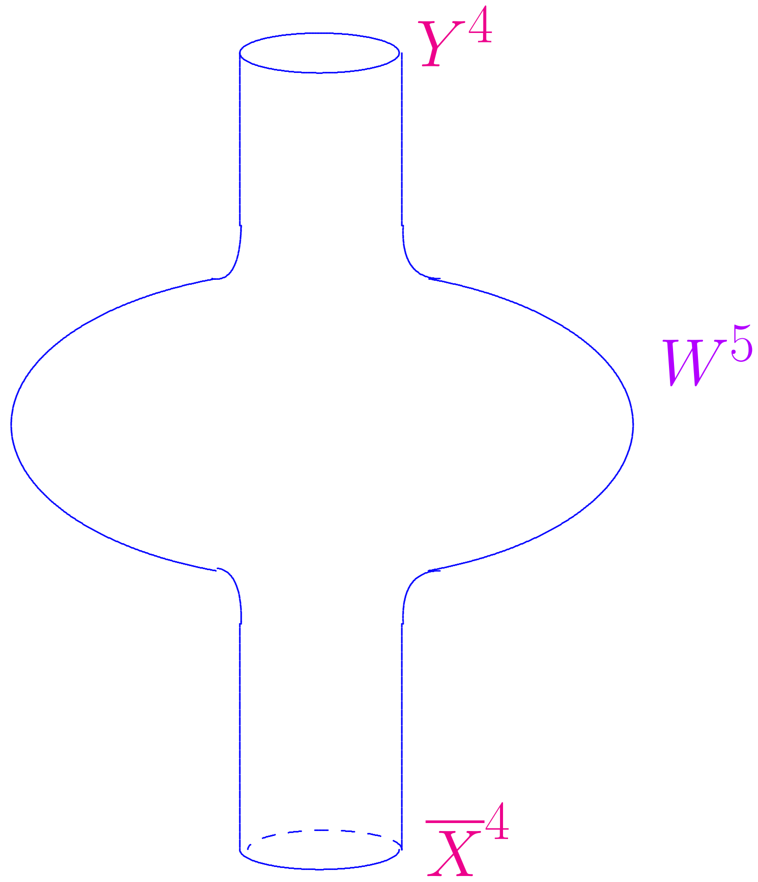
But Smale doesn't apply when $n = 4$!



However, Freedman was able to prove:



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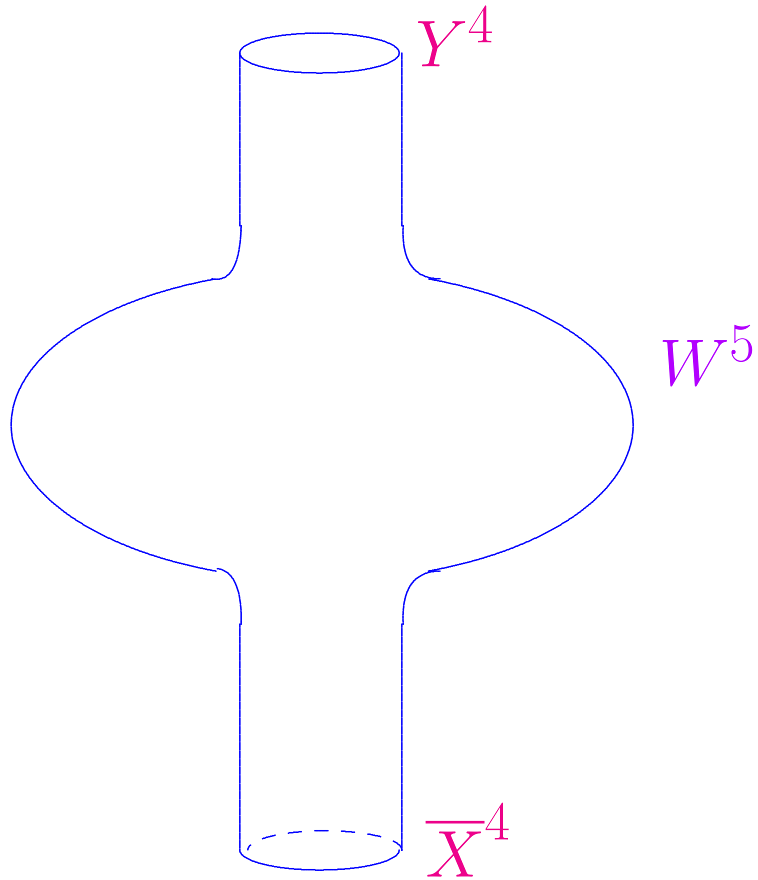
If $\pi_1 = 0$, then W^5 is homeomorphic to product,



However, Freedman was able to prove:

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so X^4 is homeomorphic to Y^4.

Theorem (Freedman/Donaldson). *Two smooth compact simply connected oriented* 4*-manifolds are orientedly homeomorphic if and only if*

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Warning: "Exotic differentiable structures!"

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Typically, one homeotype $\longleftrightarrow$ ∞ many diffeotypes.

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Corollary. *Any smooth compact* simply connected *non-spin 4-manifold M is homeomorphic to a connect sum*

$$j\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2} = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{j} \# \underbrace{\overline{\mathbb{CP}_2} \# \cdots \# \overline{\mathbb{CP}_2}}_{k}$$

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where $j = b_+(M)$ *and* $k = b_-(M)$.

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$K3$ manifold...

$K3$ = Kummer-Kähler-Kodaira surface.

—André Weil

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"...et de la belle montagne K2 au Cachemire."

—André Weil, 1958

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Simply connected complex surface with $c_1 = 0$.

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Spin, $\chi = 24$, $\tau = -16$.

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Intersection form: $2\mathsf{E}_8 \oplus 3\mathsf{H}$.

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Kummer construction:

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Begin with $T^4/\mathbb{Z}_2$:

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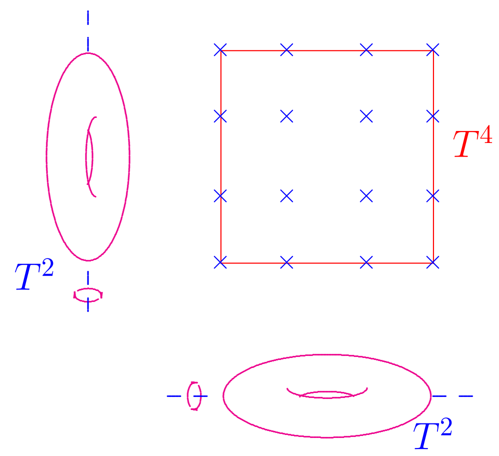

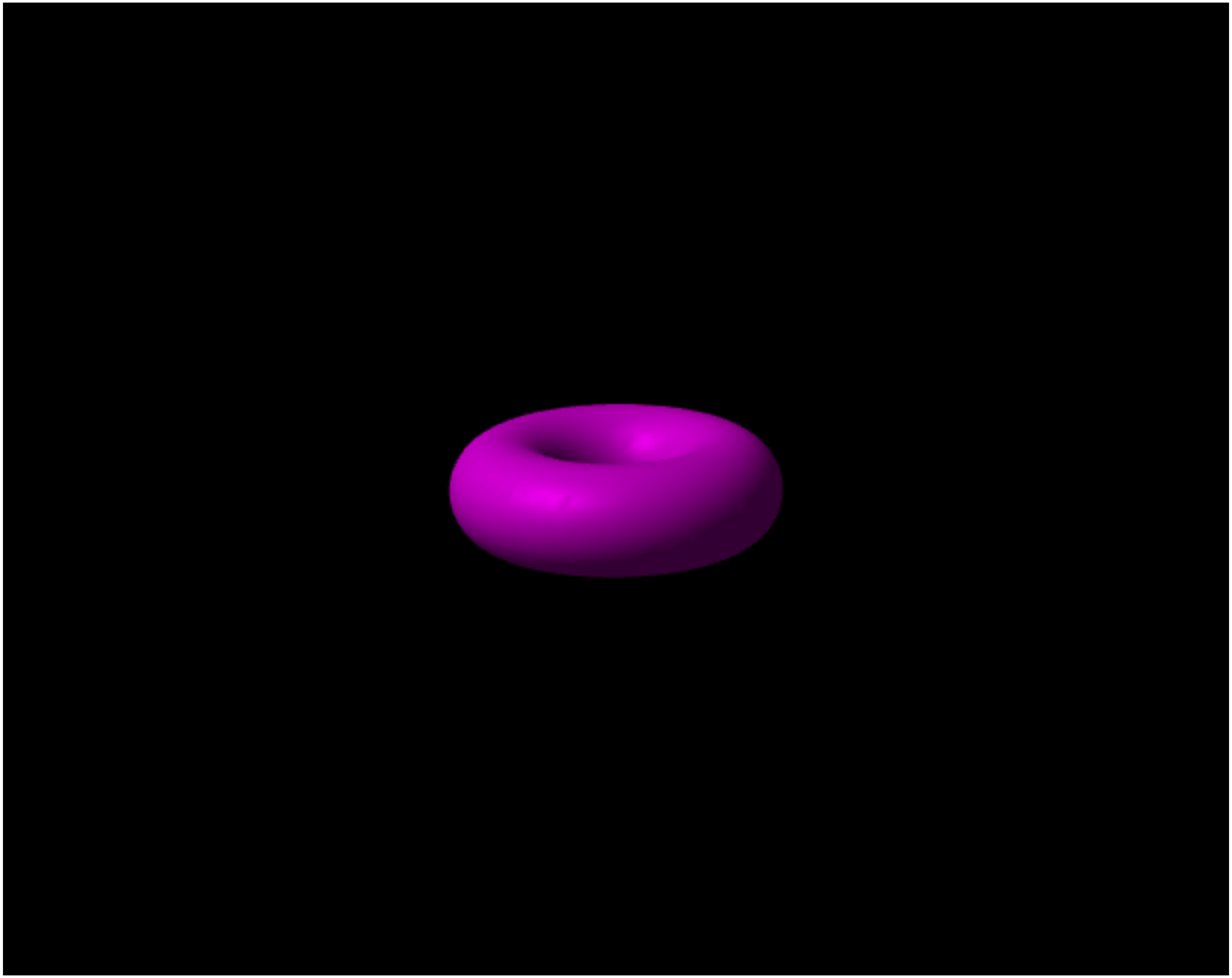

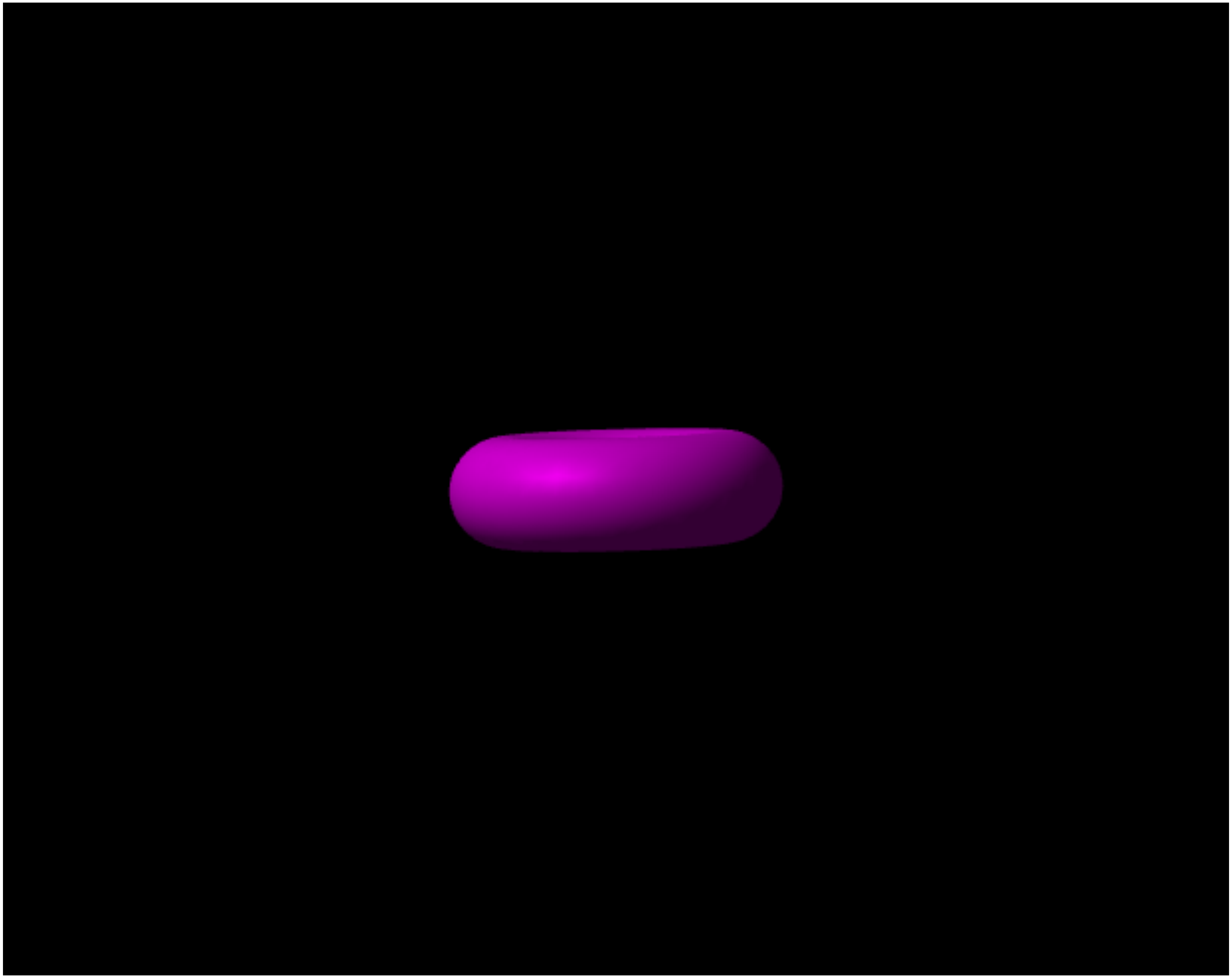

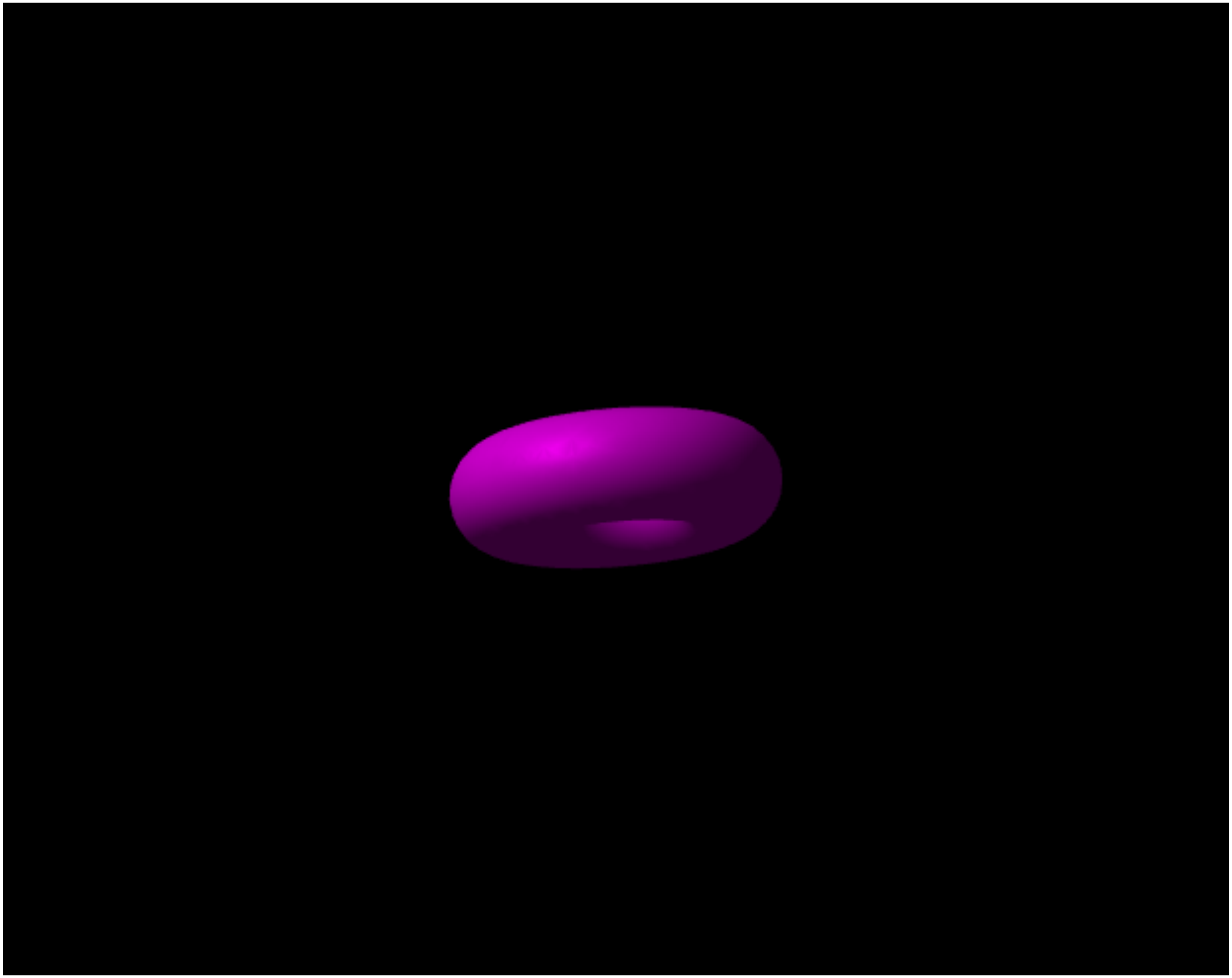

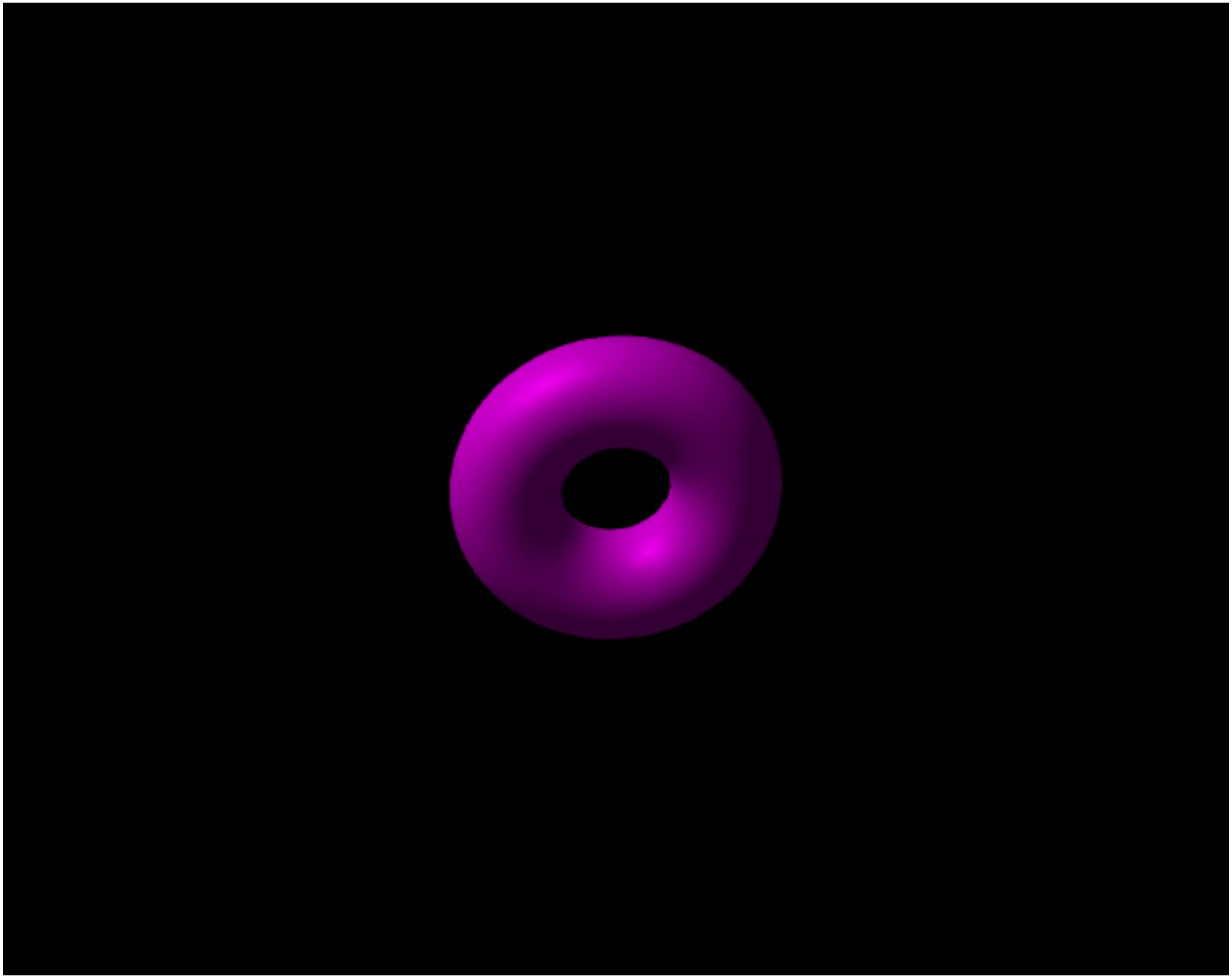

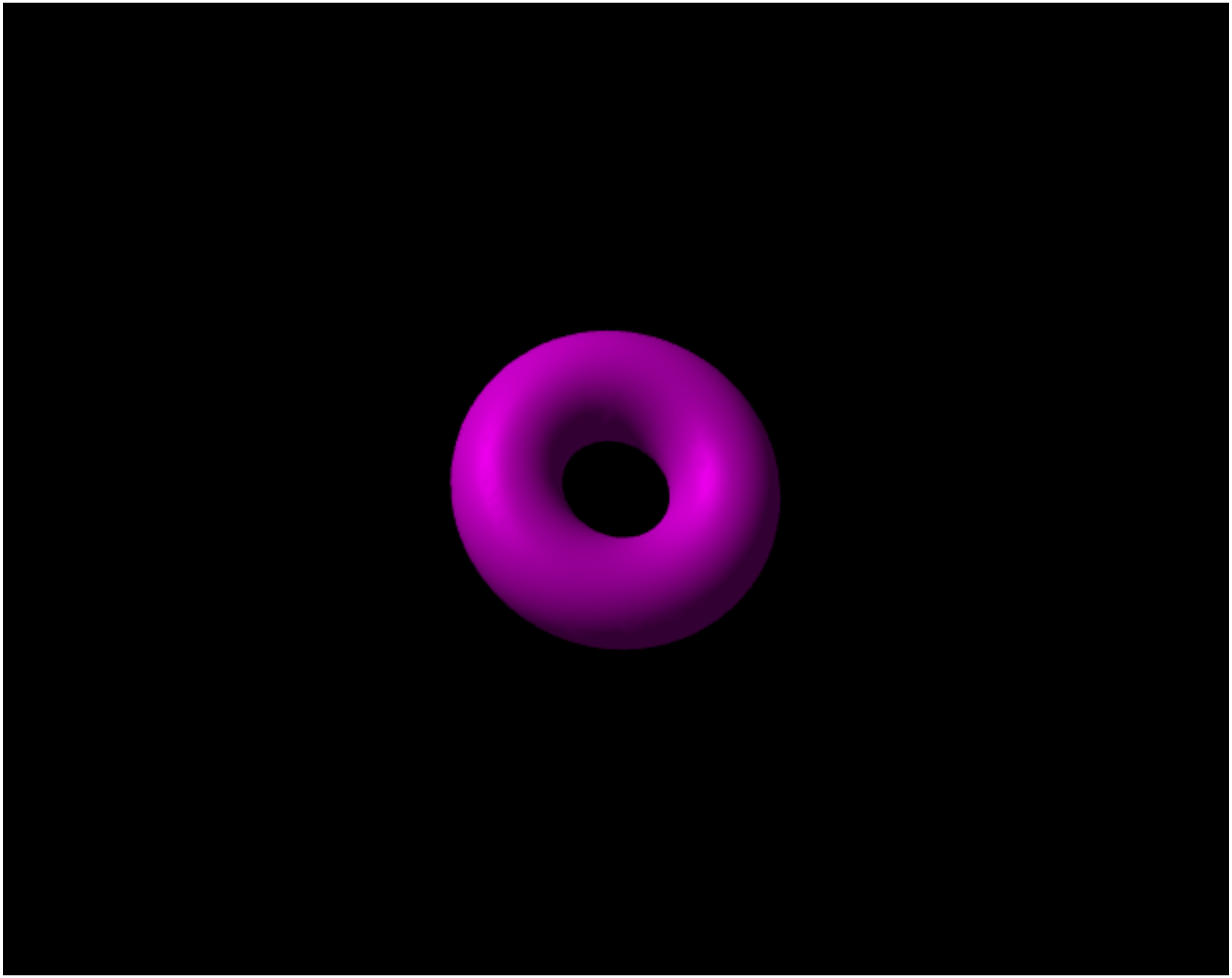

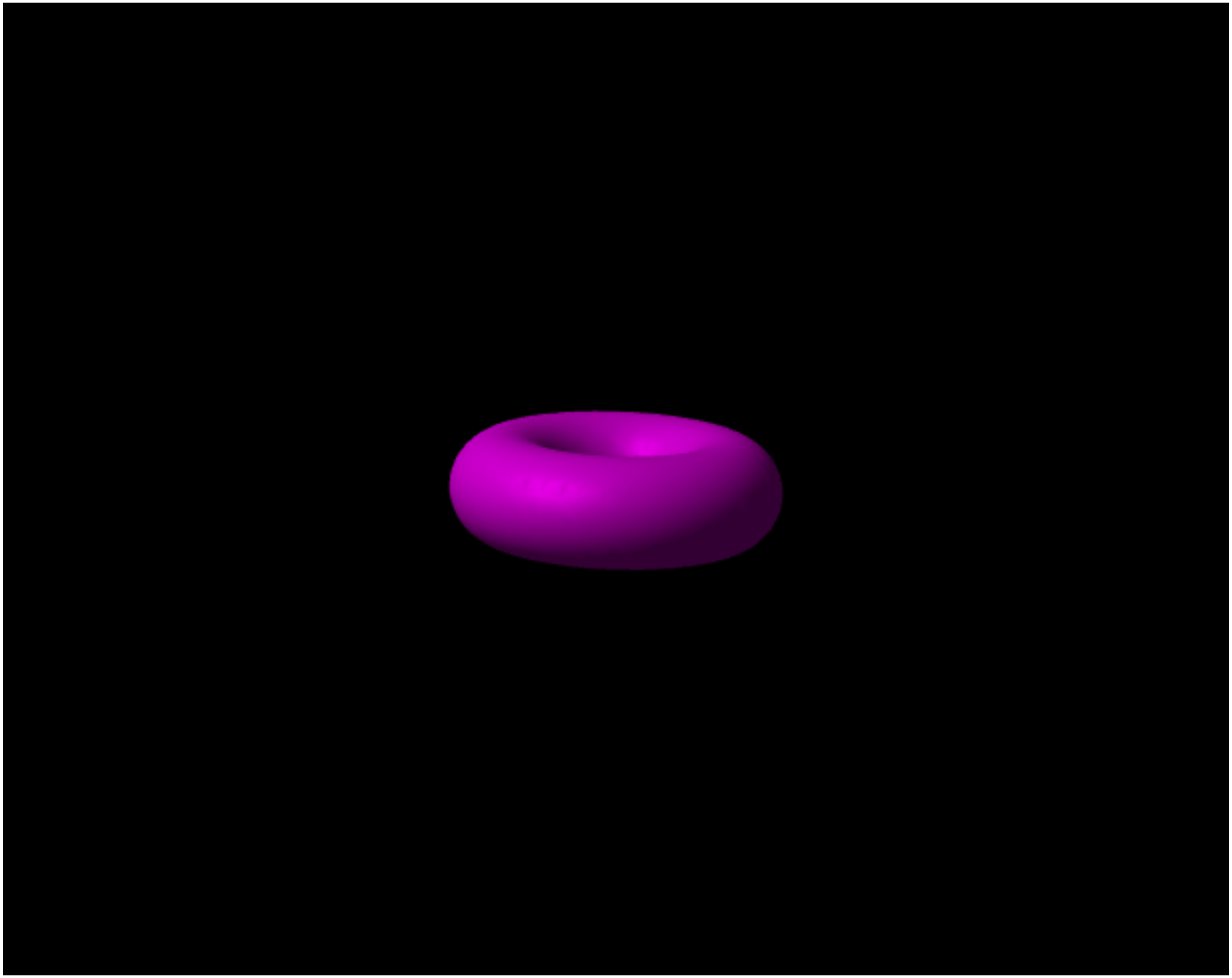

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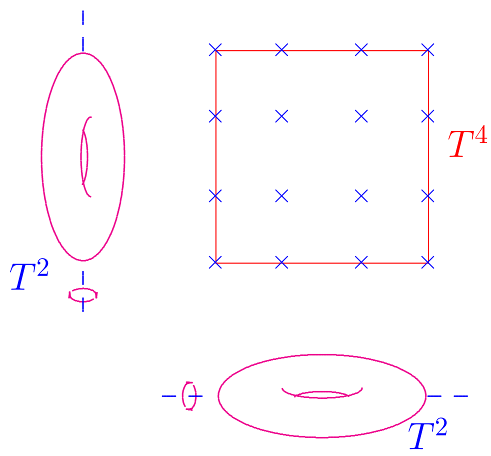
Kummer construction:

Begin with $T^4/\mathbb{Z}_2$:



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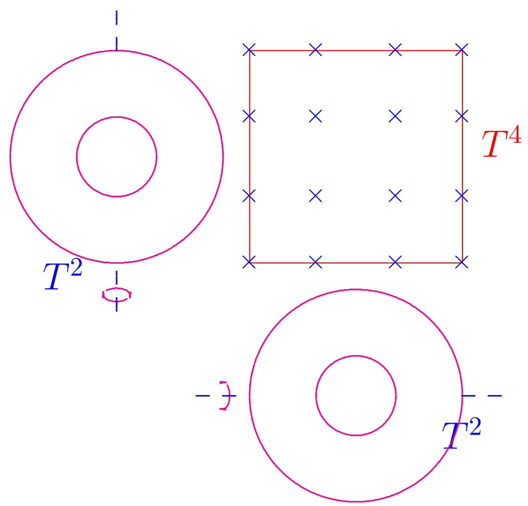

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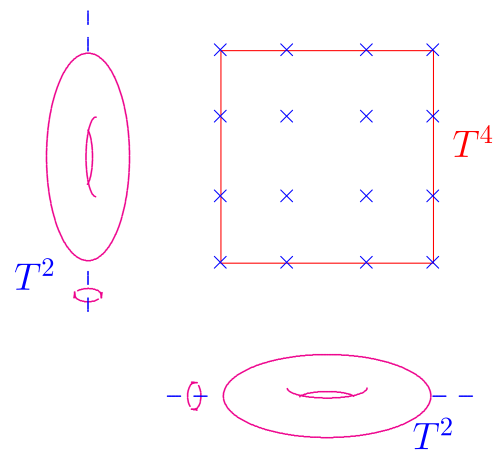

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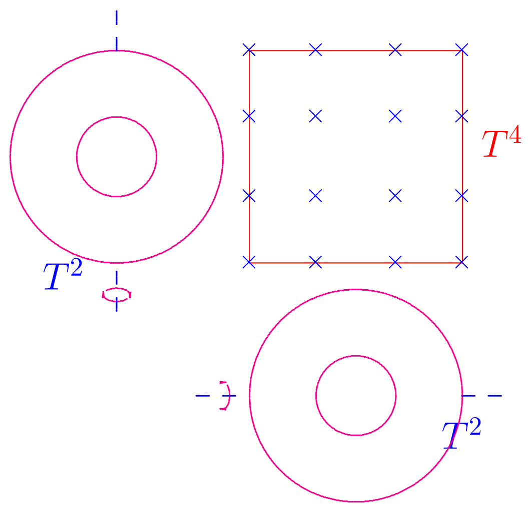

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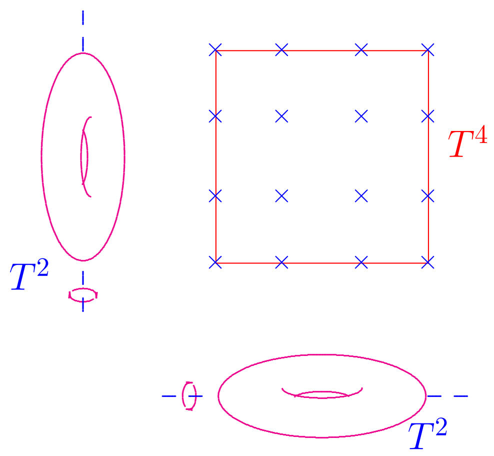

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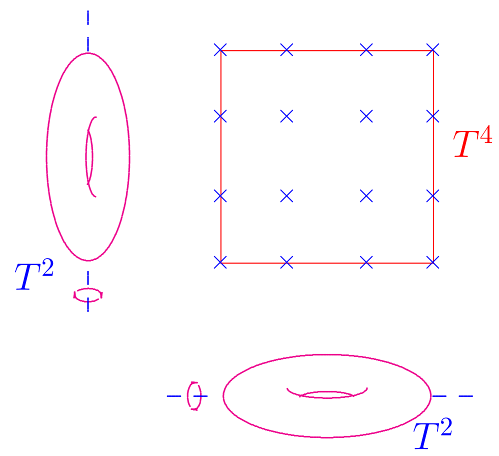

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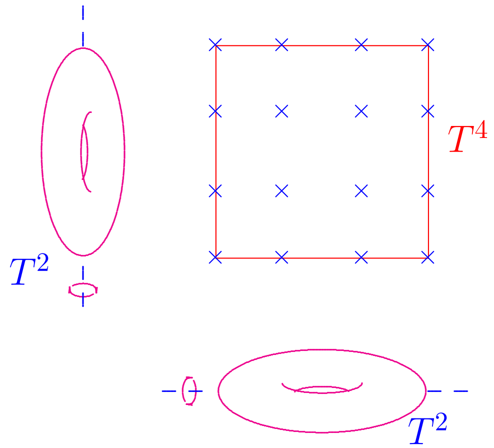

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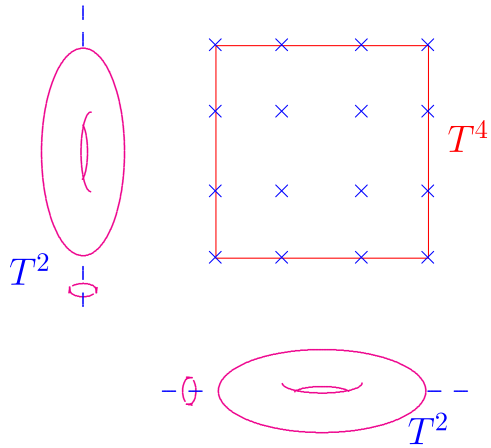

Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of T^*S^2.

Result is a $K3$ surface.

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T^4 = Picard torus of curve of genus 2.

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$$0 = (x^2 + y^2 + z^2 - w^2)^2 - 8[(1 - z^2)^2 - 2x^2][(1 + z^2)^2 - 2y^2]$$

Theorem (Freedman/Donaldson)**.** *Two smooth compact* simply connected *oriented* 4-*manifolds are orientedly homeomorphic if and only if*

- *they have the same Euler characteristic* χ*;*
- *they have the same signature* τ*; and*
- *both are spin, or both are non-spin.*

Corollary. *Any smooth compact* simply connected *non-spin* 4-*manifold* M *is homeomorphic to a connect sum* $j\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$.

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Best currently known result is by Furuta:

$$b_2 \geq \frac{10}{8}|\tau| + 2.$$

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Certainly true of all examples in these lectures!

Intermission

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Λ^+ self-dual 2-forms

Λ^- anti-self-dual 2-forms

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More on this in a moment!

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with equality $\Longleftrightarrow$ ∇ *is flat.*

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right)$$

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This brings us to a discussion of holonomy!

(M^n, g): holonomy

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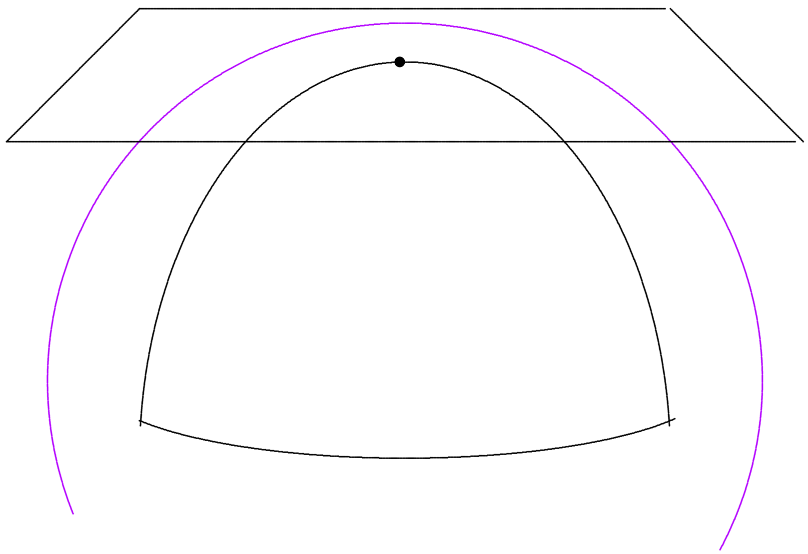

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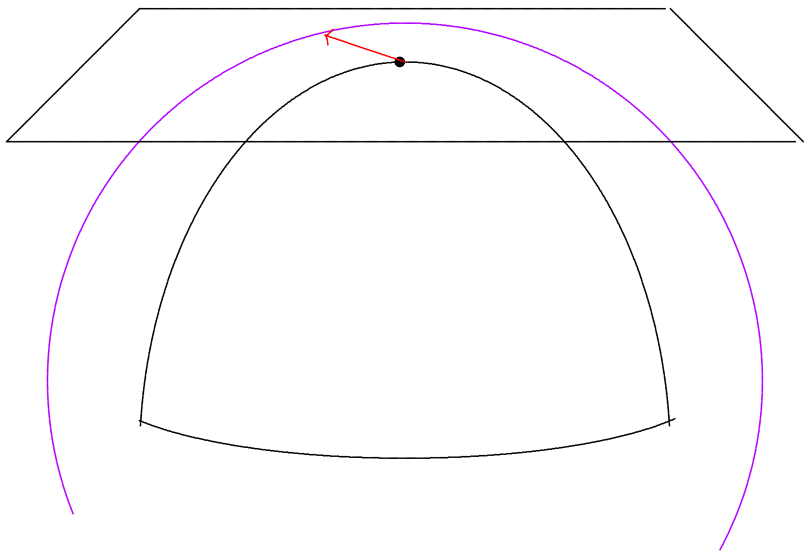

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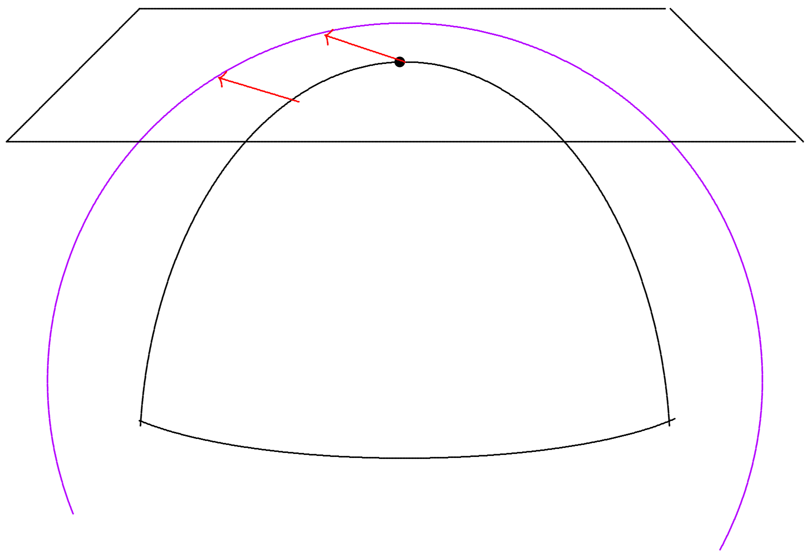

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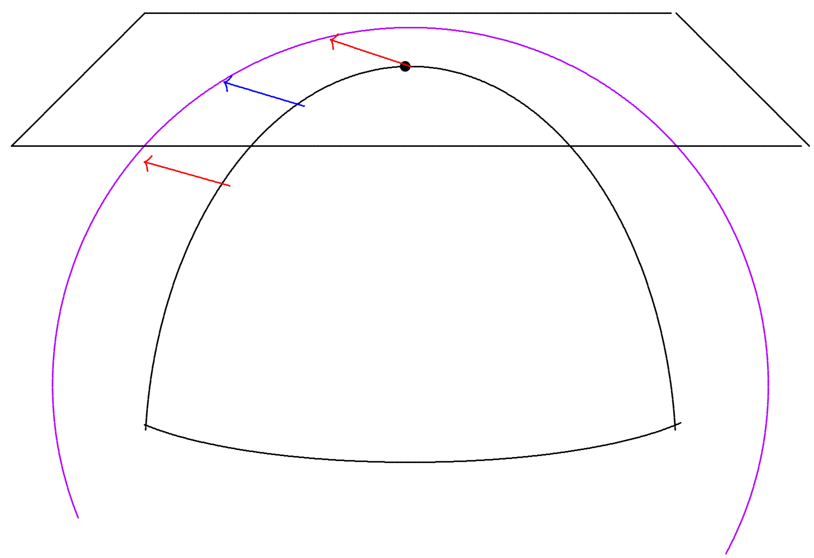

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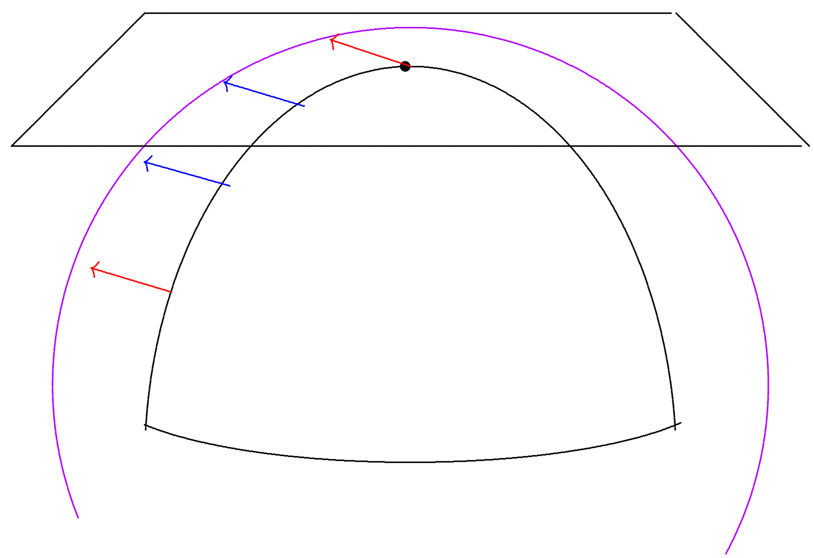

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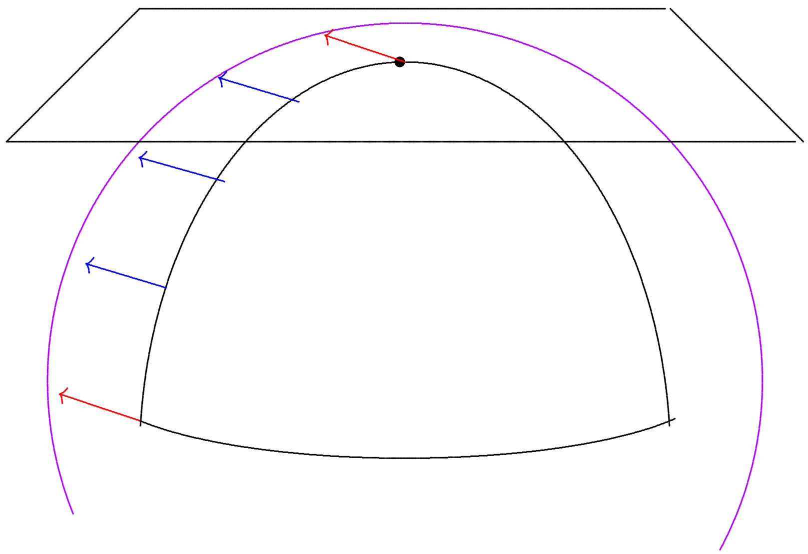

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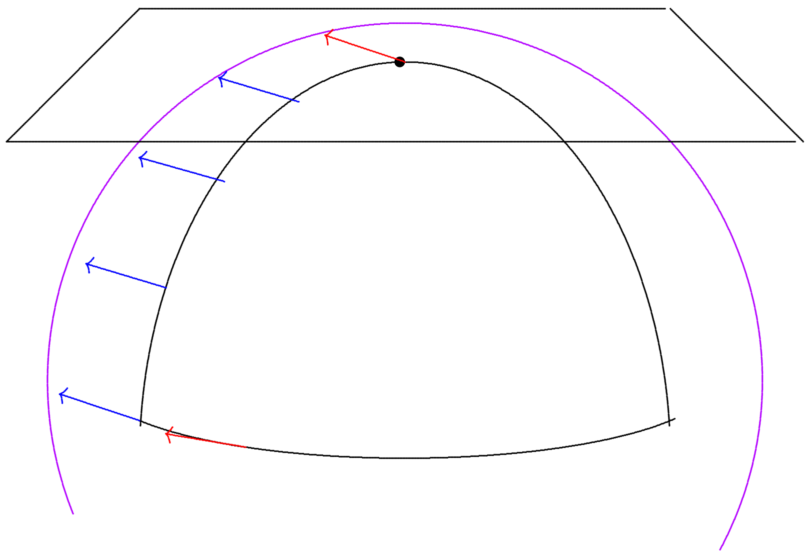

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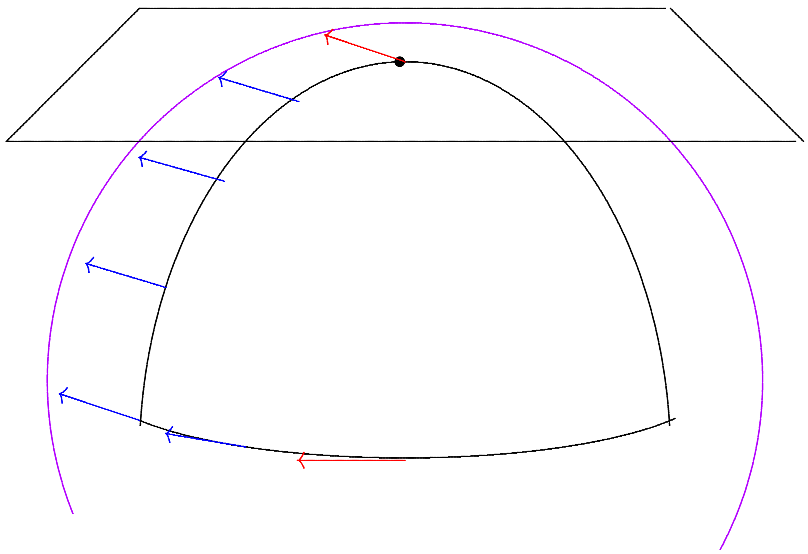

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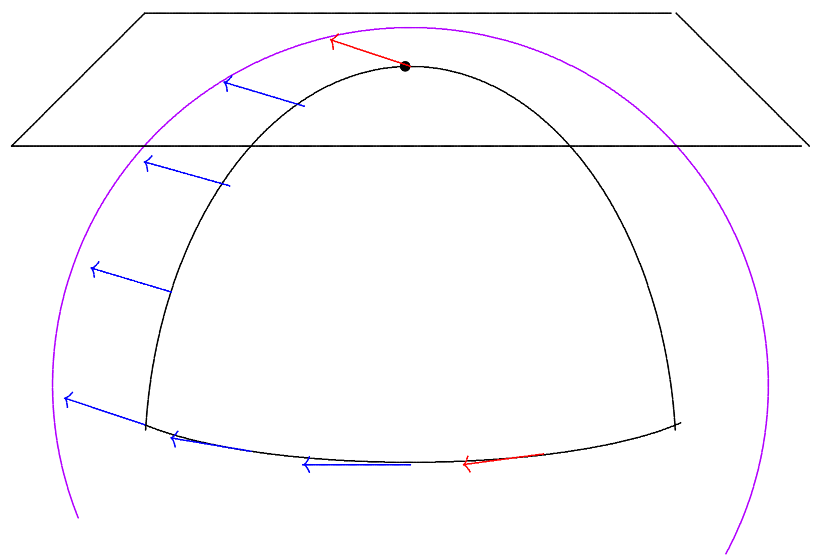

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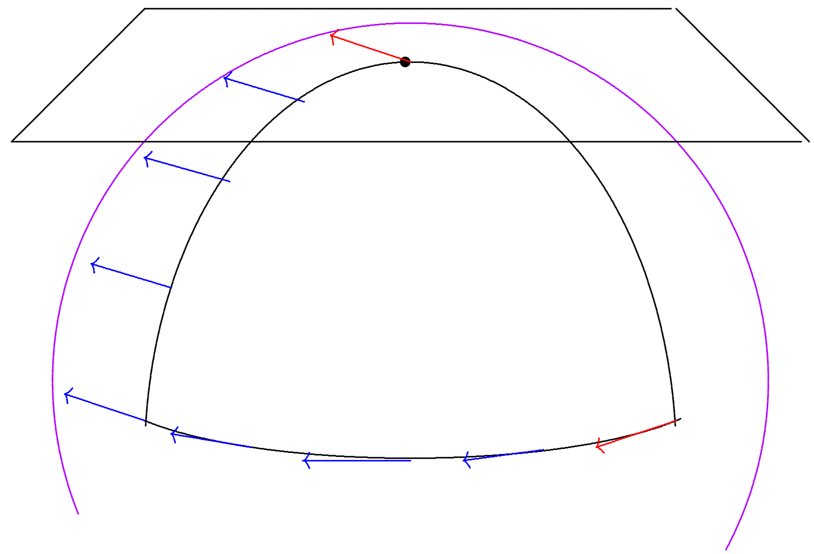

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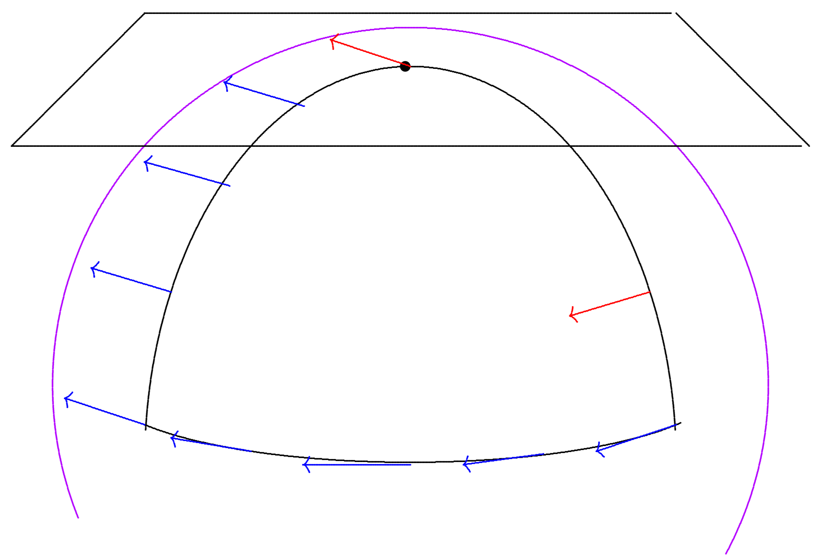

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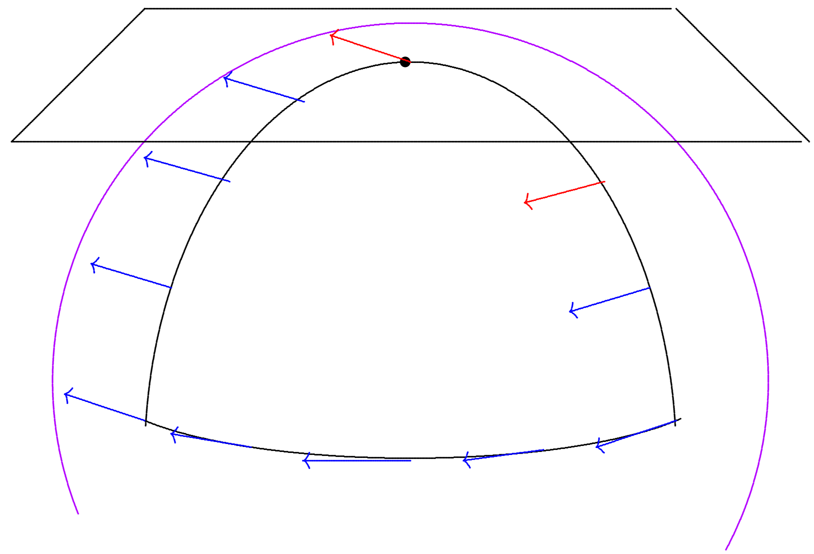

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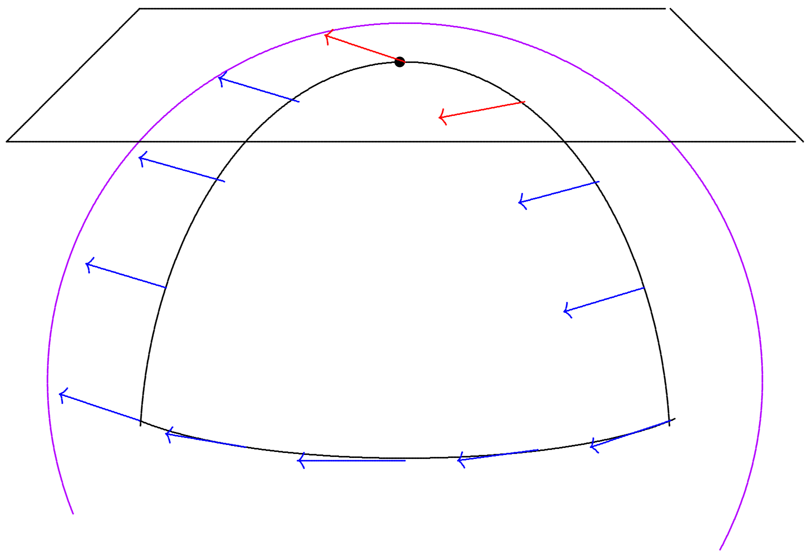

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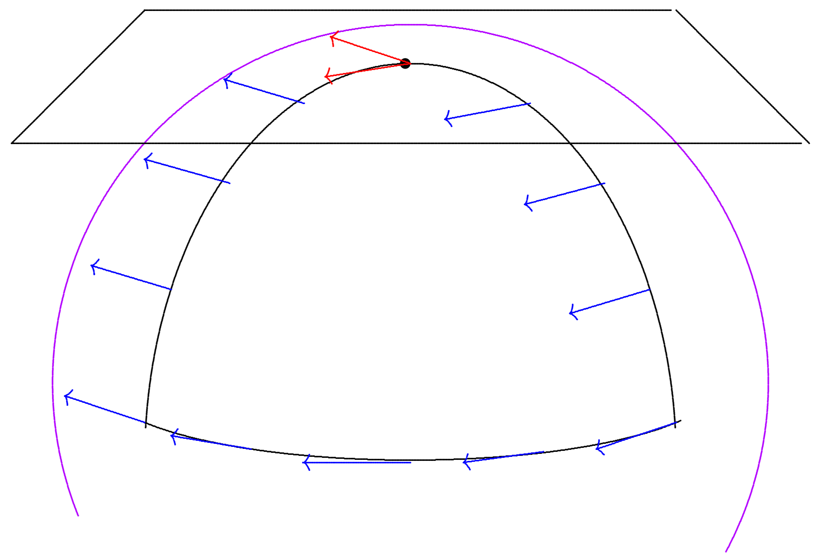

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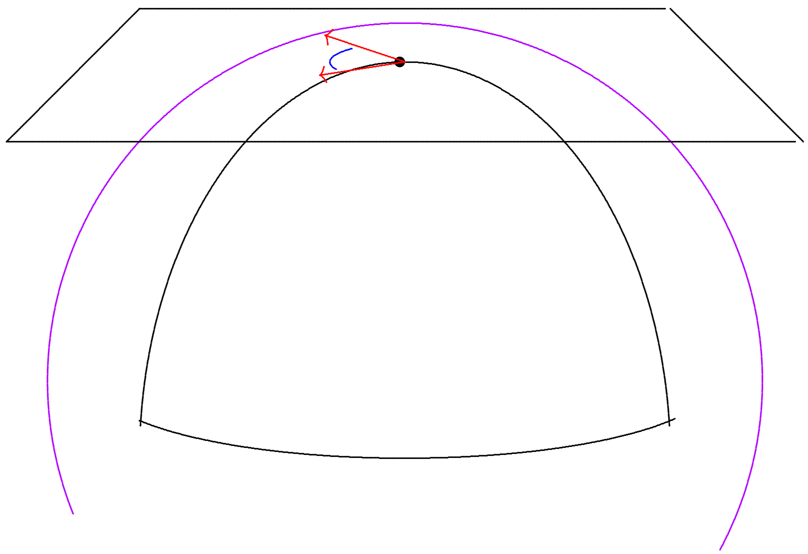
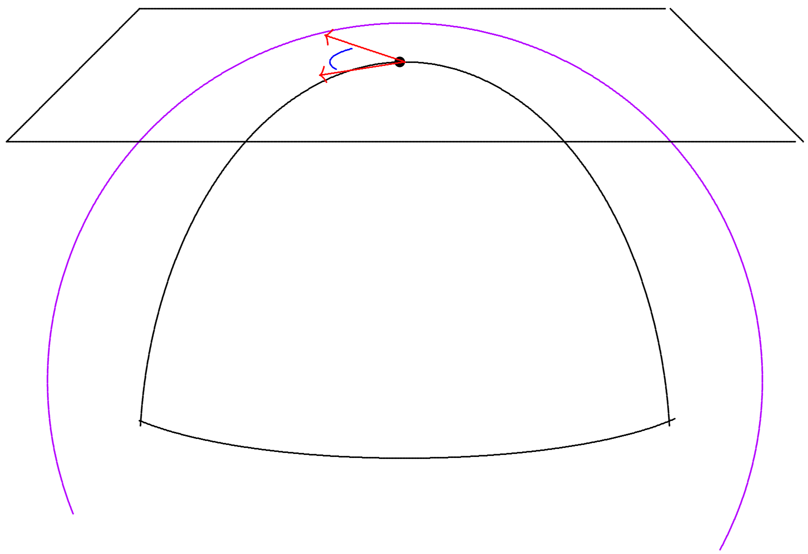


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

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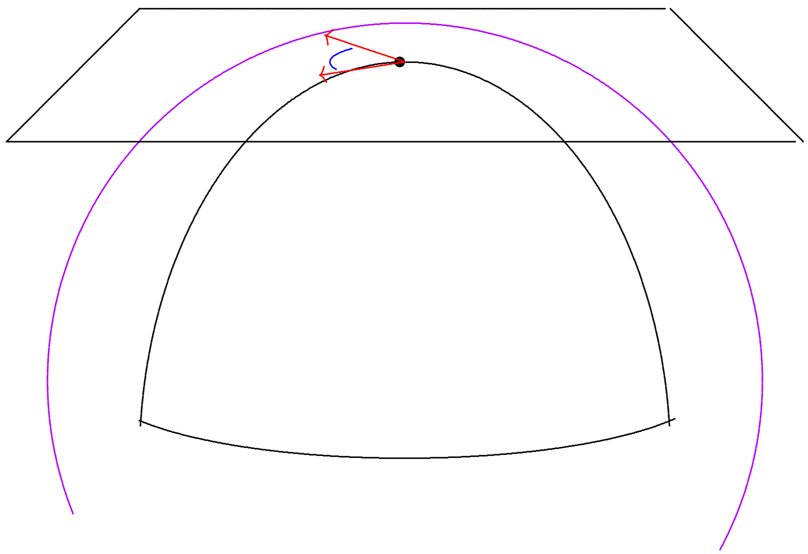

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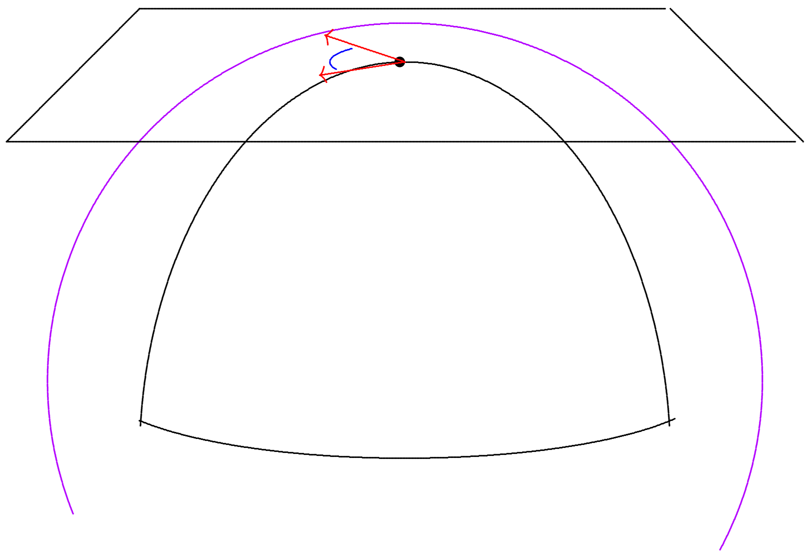

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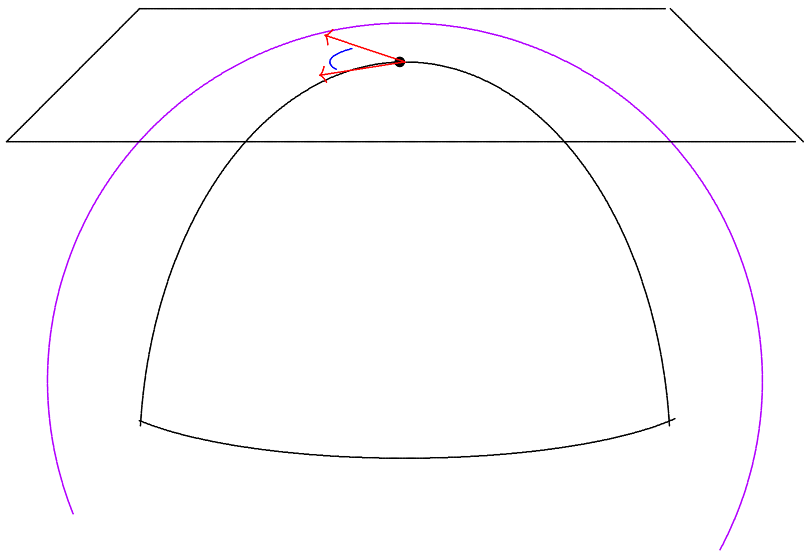

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$$J : TM \to TM, \qquad J^2 = -\mathbf{I}$$

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$\therefore$ Newlander-Nirenberg theorem $\Longrightarrow$ $\exists$ local complex coordinates $z^j = x^j + iy^j$ in which

$$J = dx^j \otimes \frac{\partial}{\partial y^j} - dy^j \otimes \frac{\partial}{\partial x^j}$$

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"Kähler class"

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The 2-form

$$ir(J\cdot, \cdot)$$

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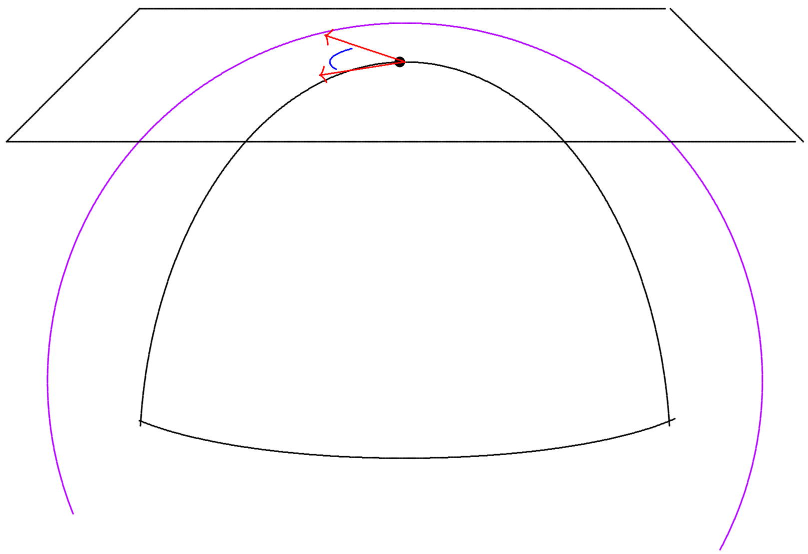

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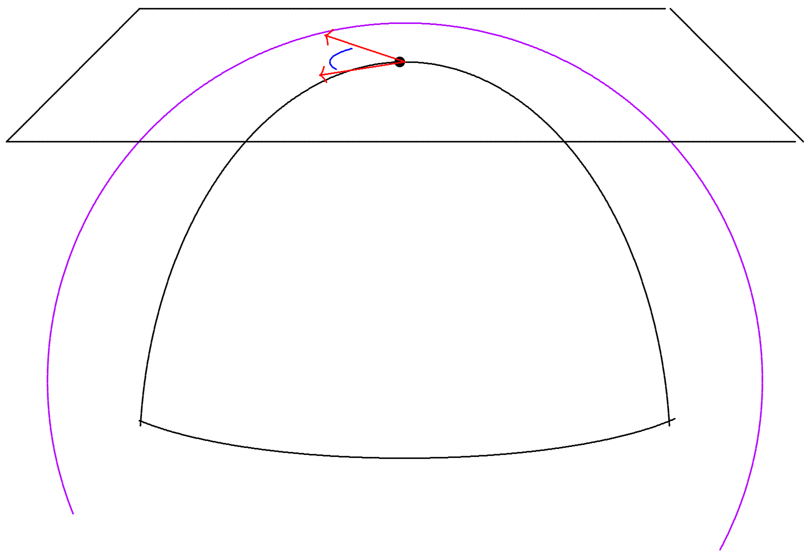

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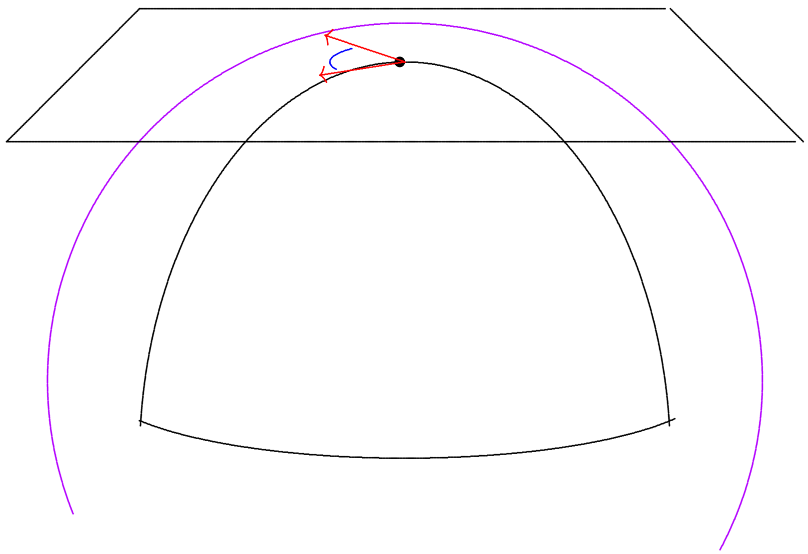

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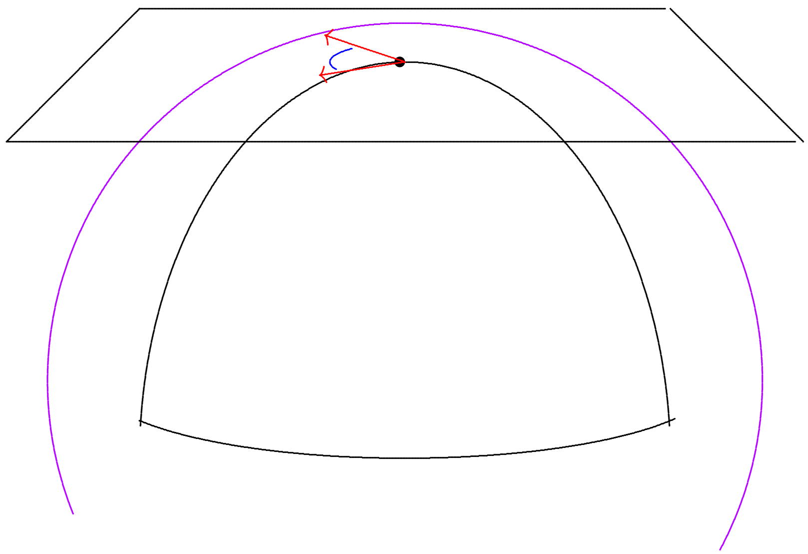
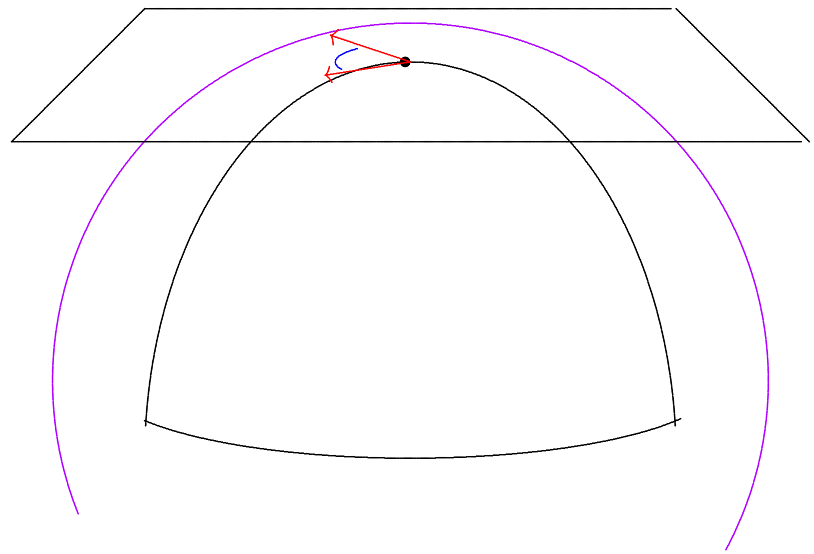


$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

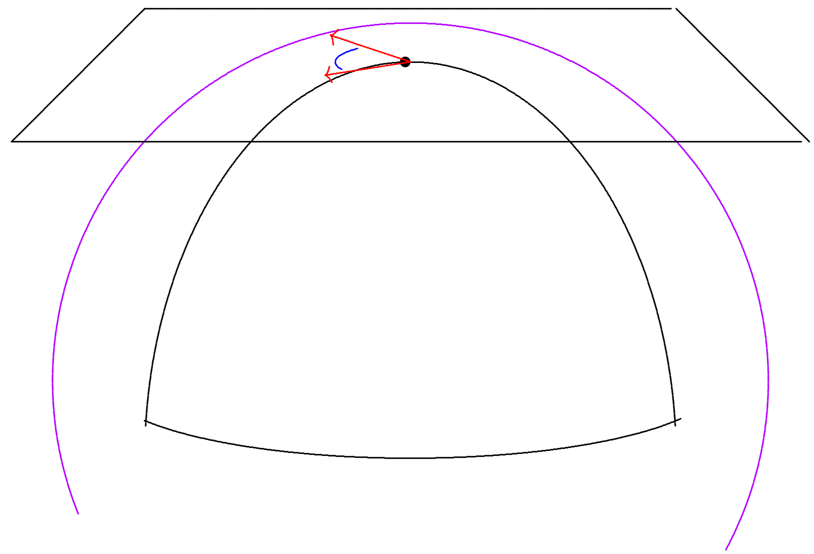
Ricci-flat Kähler metrics:

(M^{2m}, g): holonomy



Ricci-flat Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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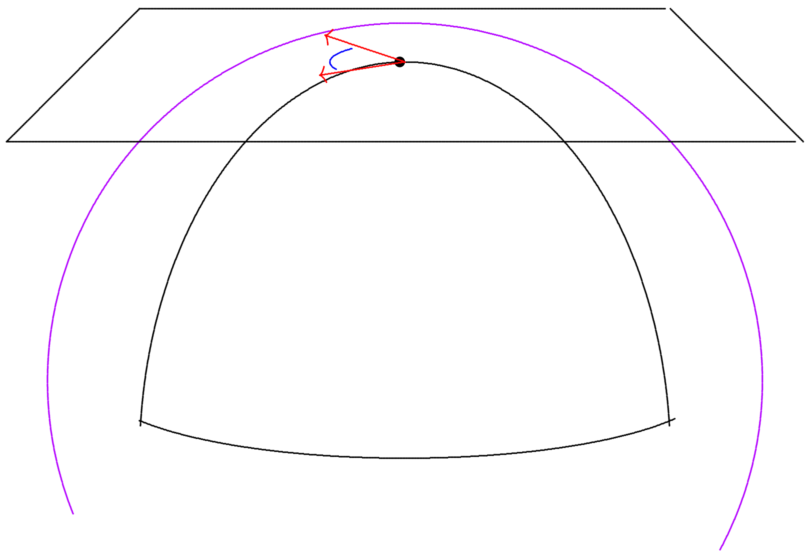

$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\{A \mid \det A = 1\}$

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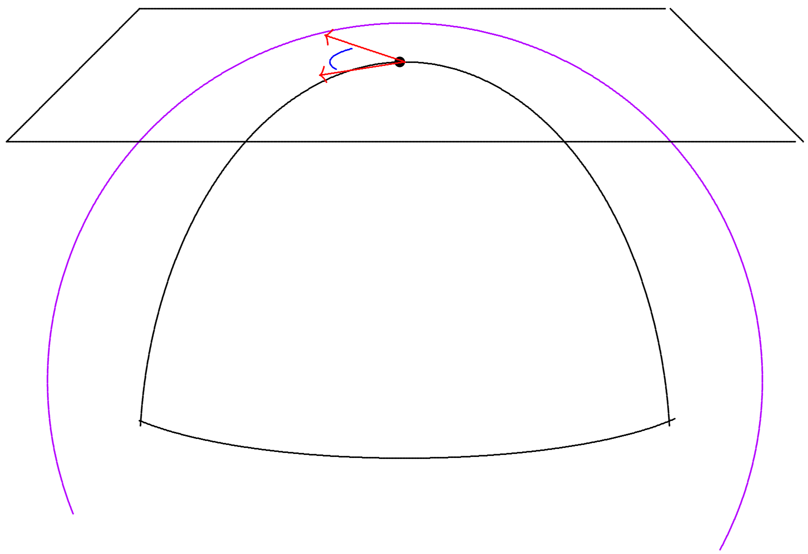

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Ricci-flat $\Longleftrightarrow$ $K = \Lambda^{m,0} = \det(\Lambda^{1,0})$ is flat.

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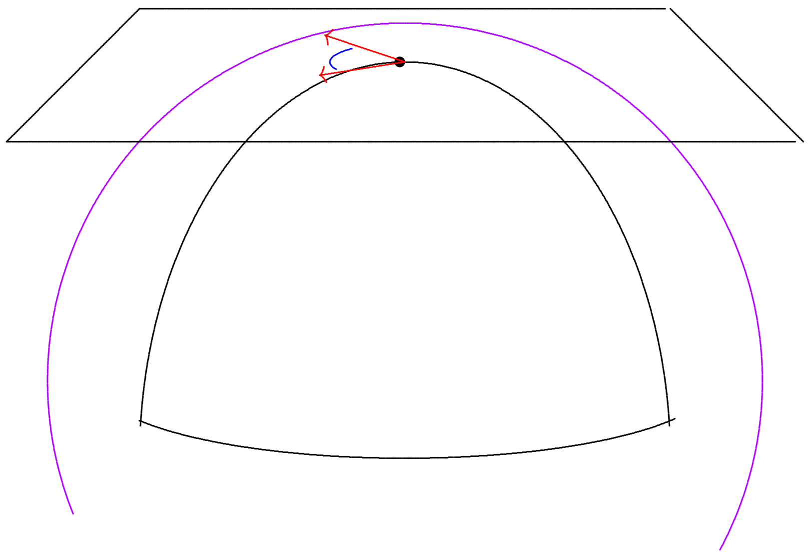

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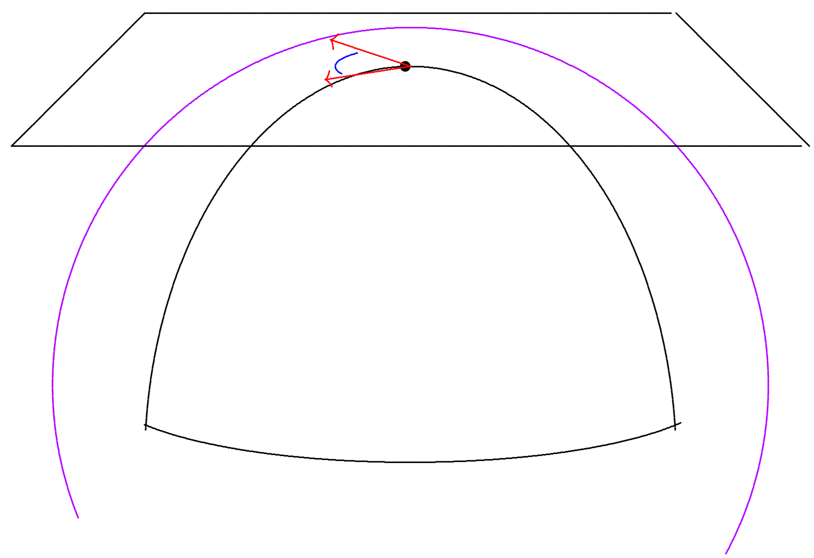
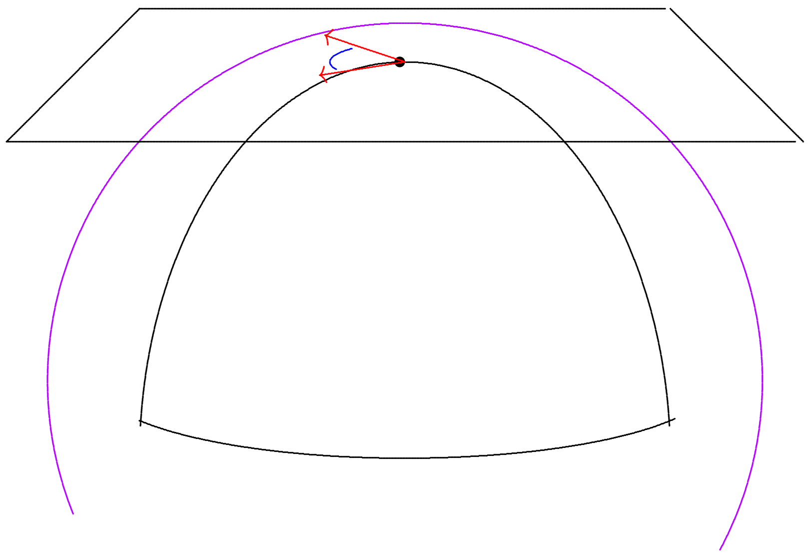


if M is simply connected.

Calabi-Yau metrics:

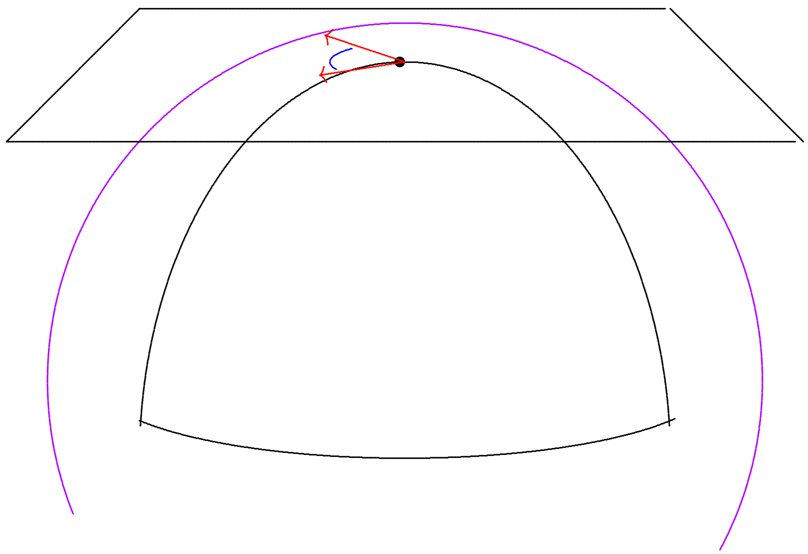
(M^{2m}, g): Calabi-Yau $\Longleftrightarrow$ holonomy $\subset \mathbf{SU}(m)$



Hyper-Kähler metrics:

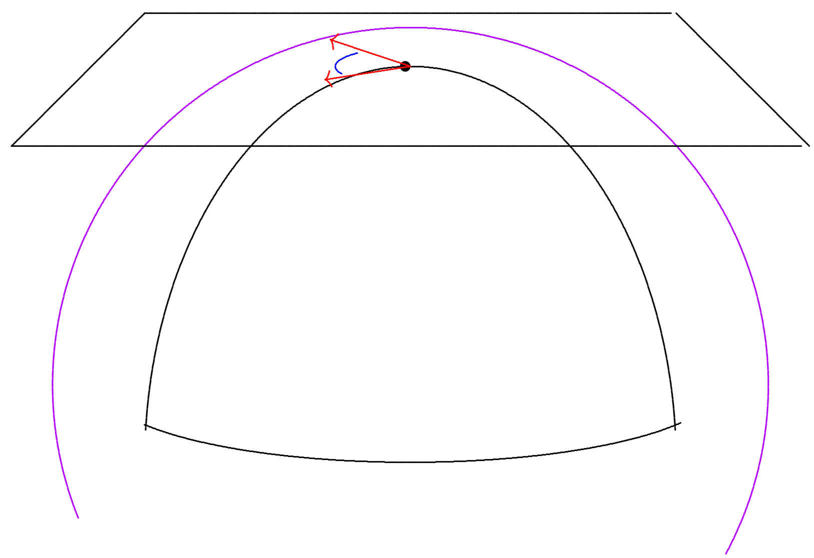
(M^{4k}, g)

holonomy



Hyper-Kähler metrics:

(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$



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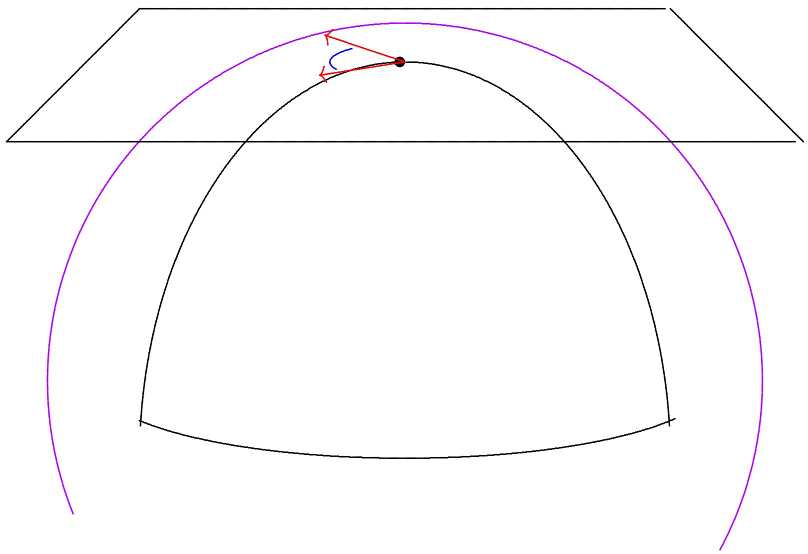

$$\mathbf{Sp}(k) := \mathbf{O}(4k) \cap \mathbf{GL}(\ell, \mathbb{H})$$

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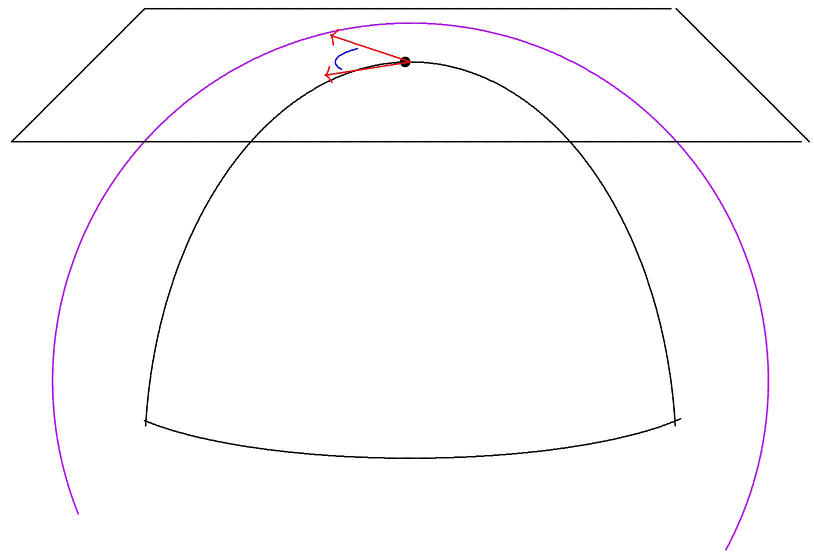

$$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$$

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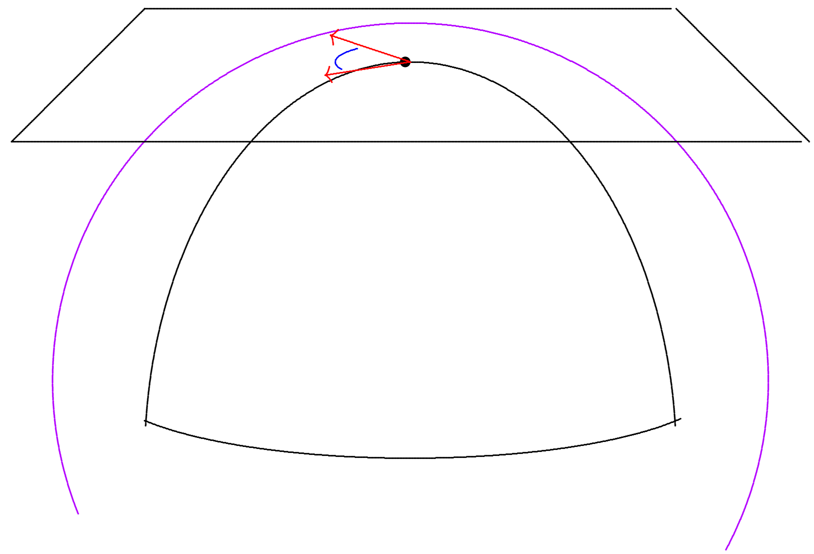

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in many ways!

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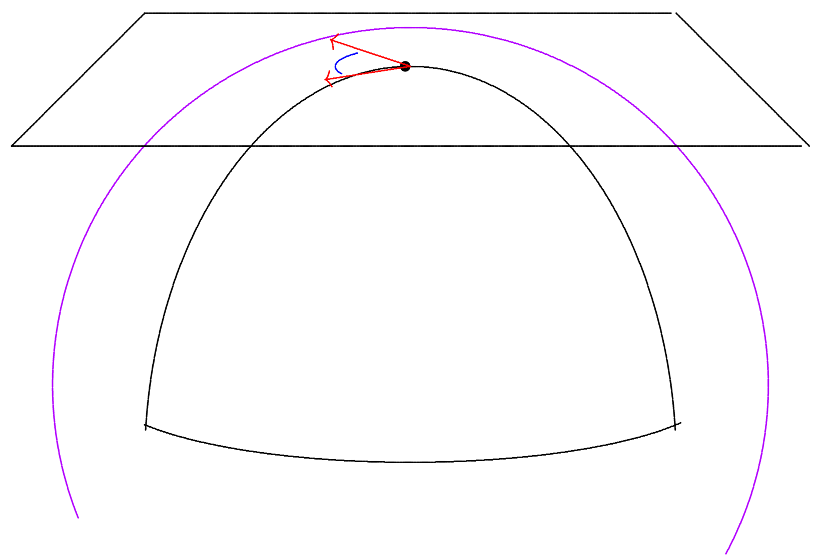

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in many ways! (For example, permute i, j, $k \ldots$)

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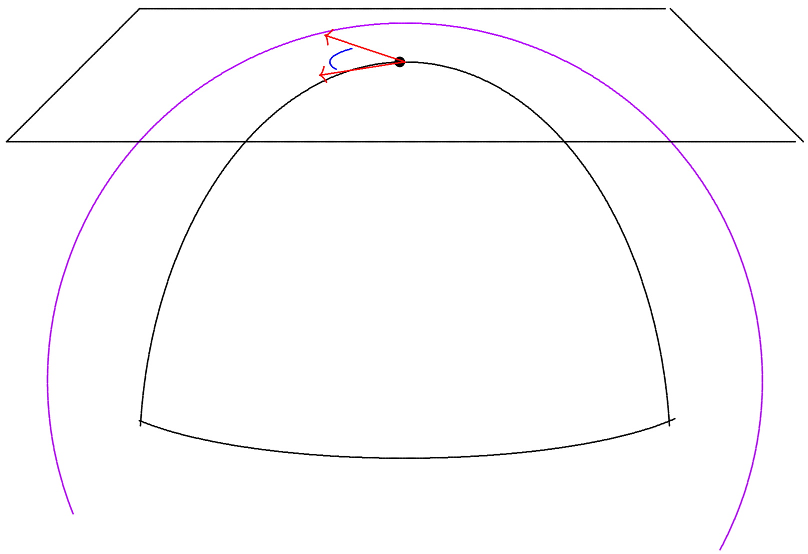

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Ricci-flat and Kähler,

for many different complex structures!

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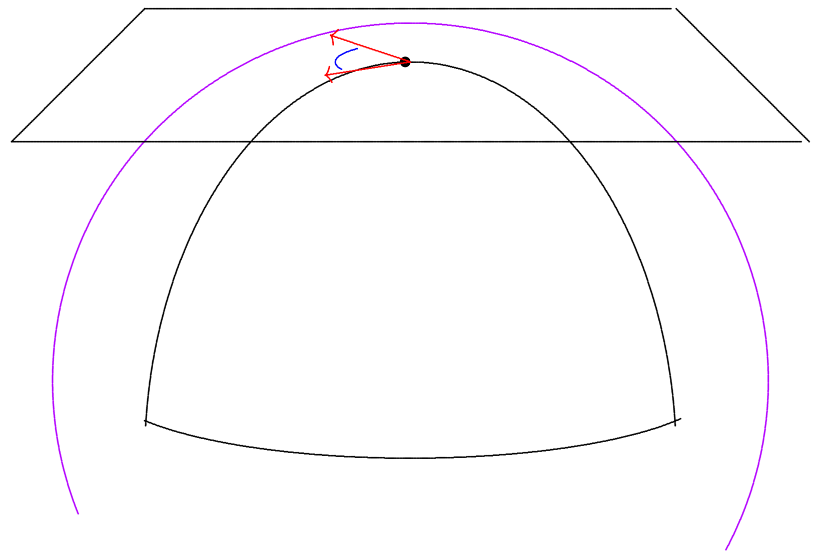
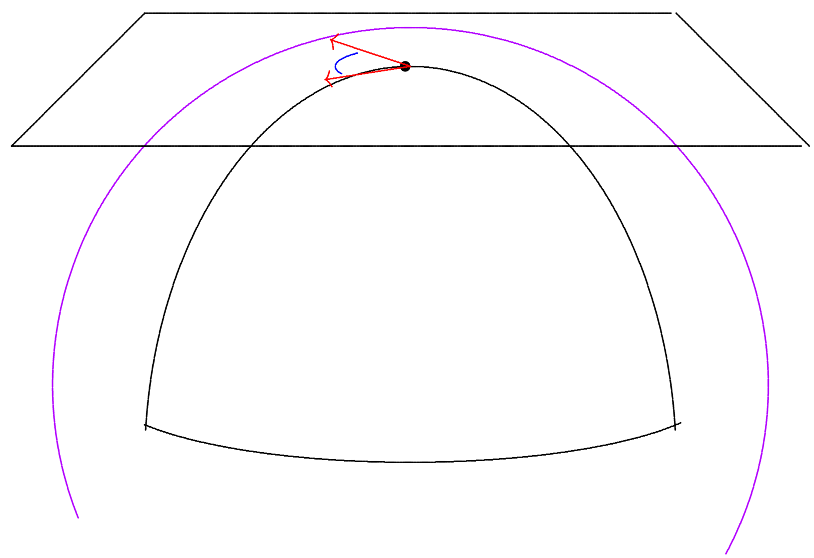


$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$\mathbf{Sp}(1) = \mathbf{SU}(2)$

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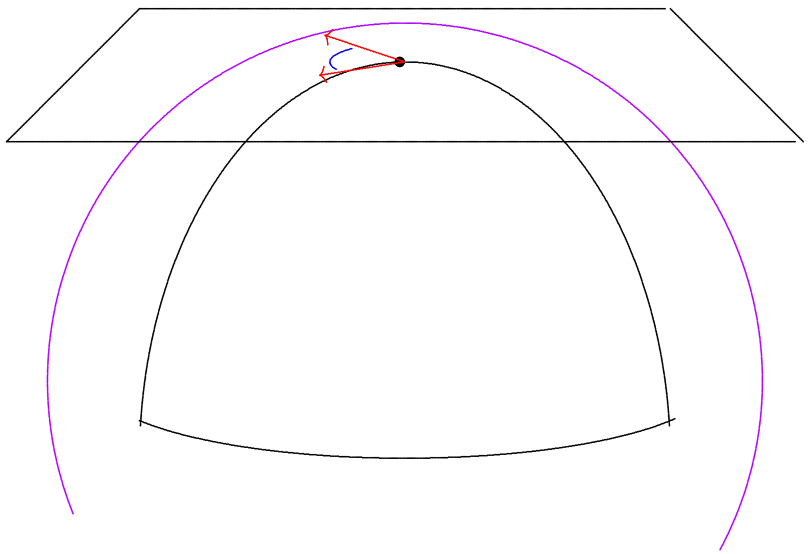

$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

For (M^4, g):

hyper-Kähler $\Longleftrightarrow$ Calabi-Yau.

Hyper-Kähler metrics:

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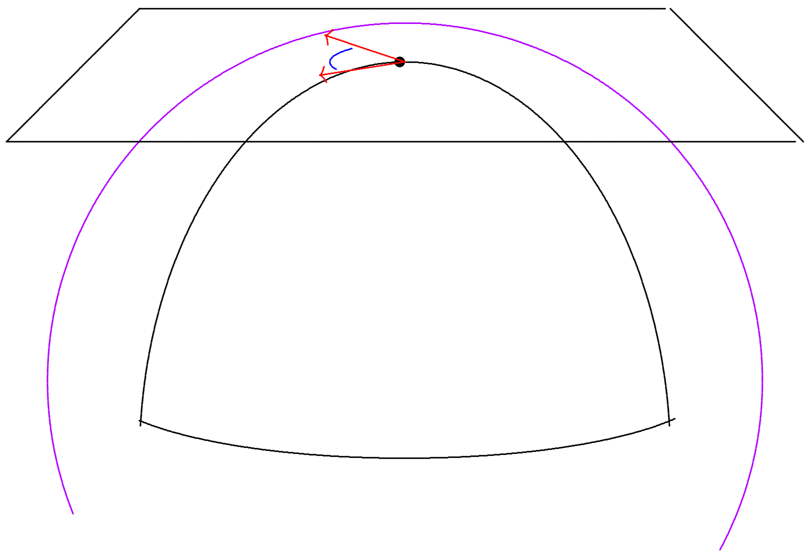

$\mathbf{Sp}(1) = \mathbf{SU}(2)$

When (M^4, g) simply connected:

hyper-Kähler $\Longleftrightarrow$ Ricci-flat Kähler.

Locally Hyper-Kähler metrics:

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We say that (M^4, g) is locally hyper-Kähler

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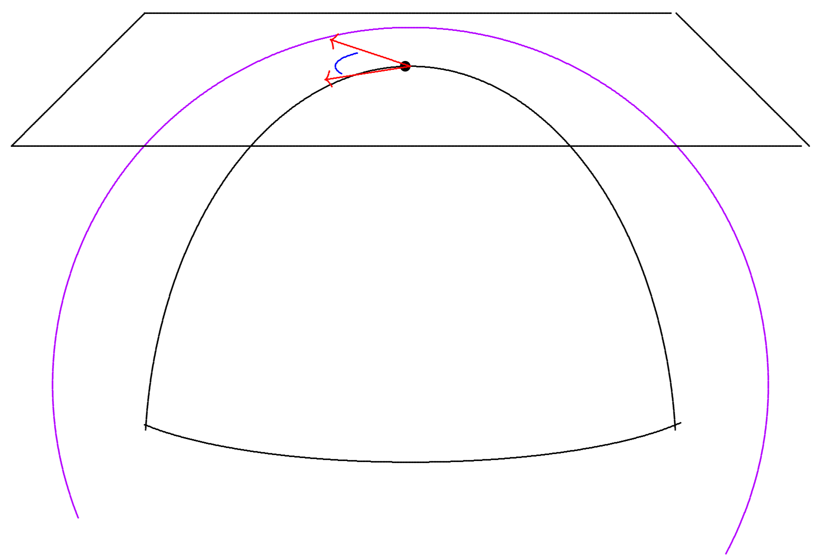

$\Longleftrightarrow$ (M^4, g) has reduced holonomy in $\mathbf{Sp}(1)$,

Locally Hyper-Kähler metrics:

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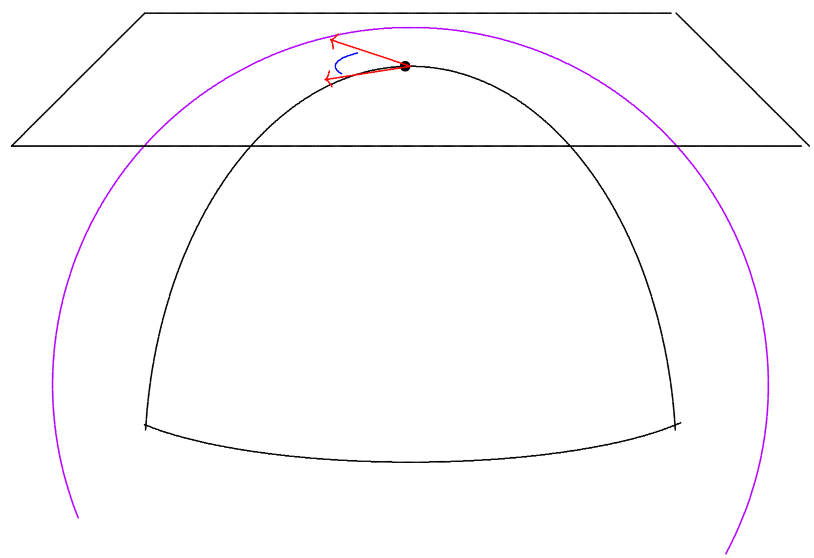

$\Longleftrightarrow$ (M^4, g) has reduced holonomy in $\mathbf{Sp}(1)$, where the reduced holonomy group is defined using only curves that represent $0 \in \pi_1(M, p)$.

Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented* M^4 *admits Einstein* g*, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if Λ^+ *is flat on* (M, g)*.*

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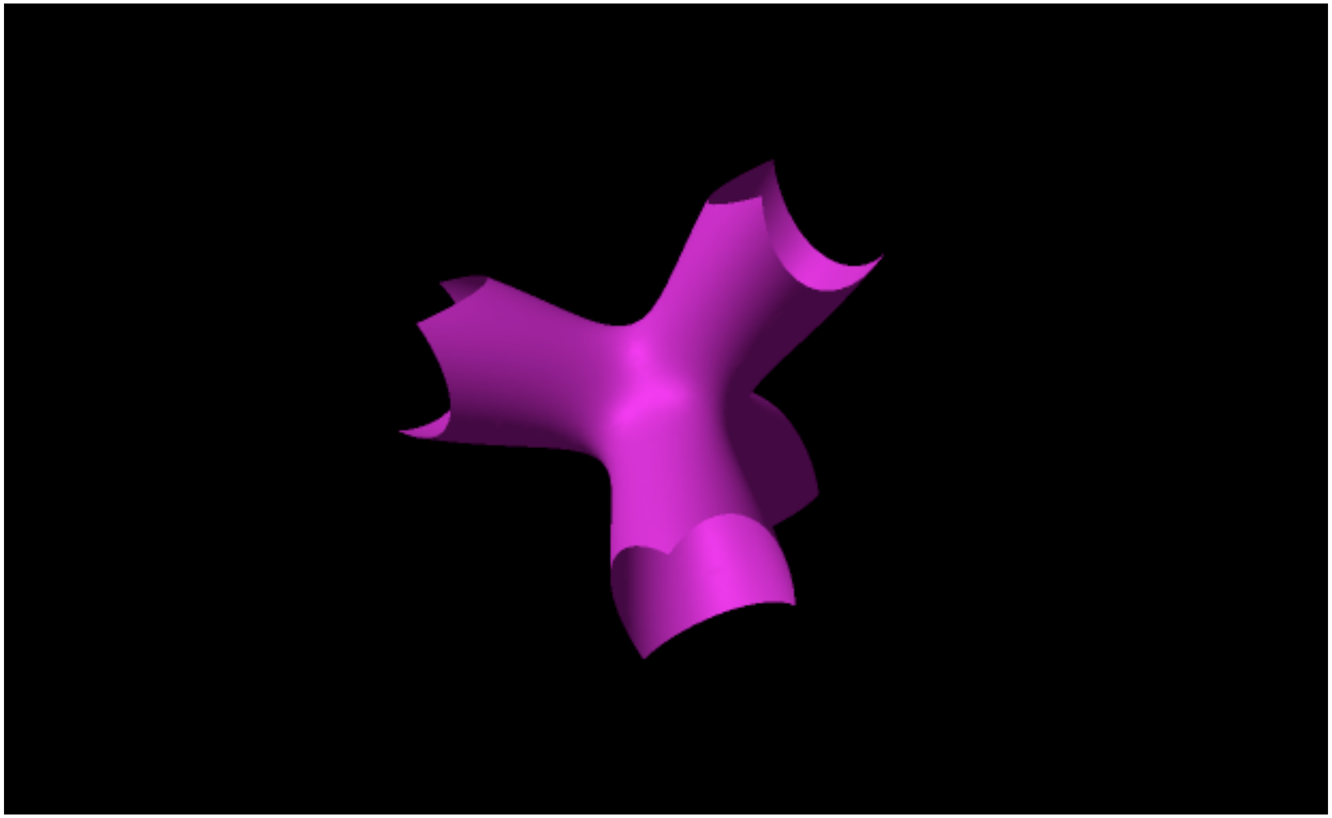

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Special Case. *If (M^4, J) is a compact complex surface, when does M^4 admit an Einstein metric g (unrelated to J)?*

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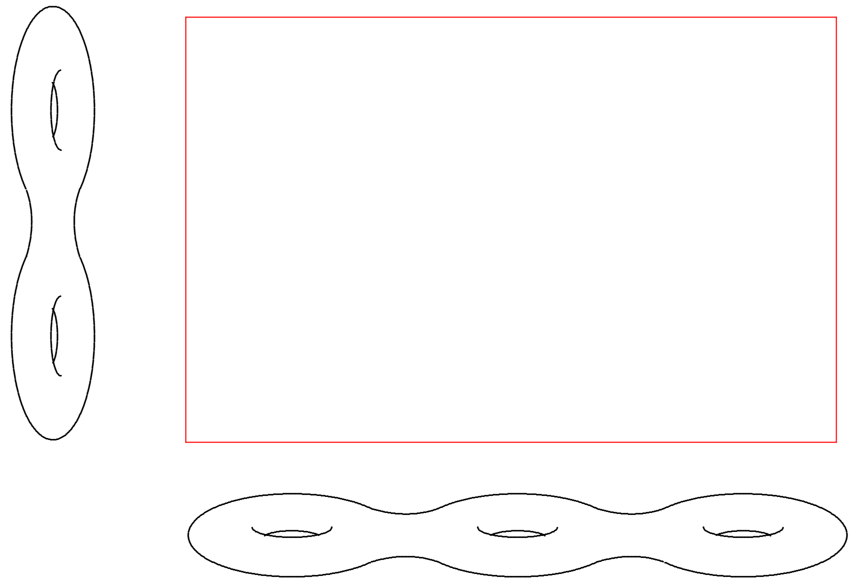

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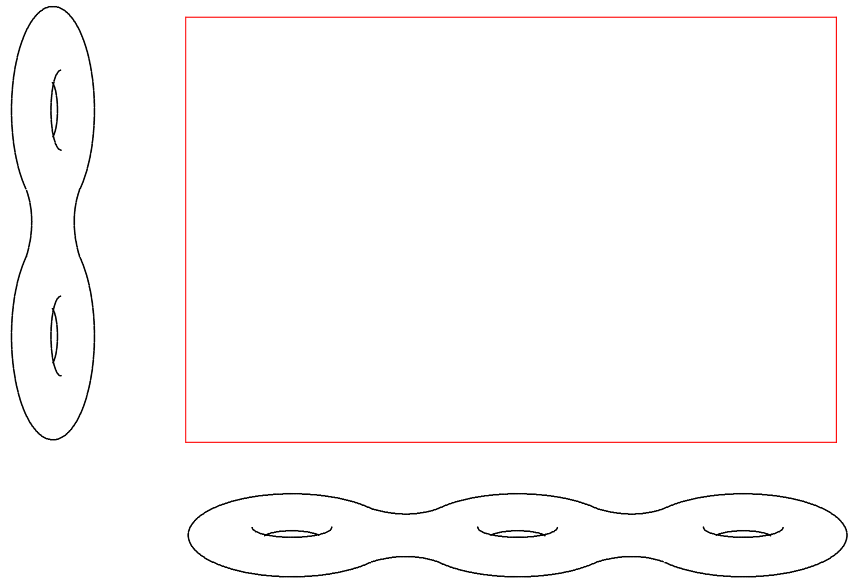

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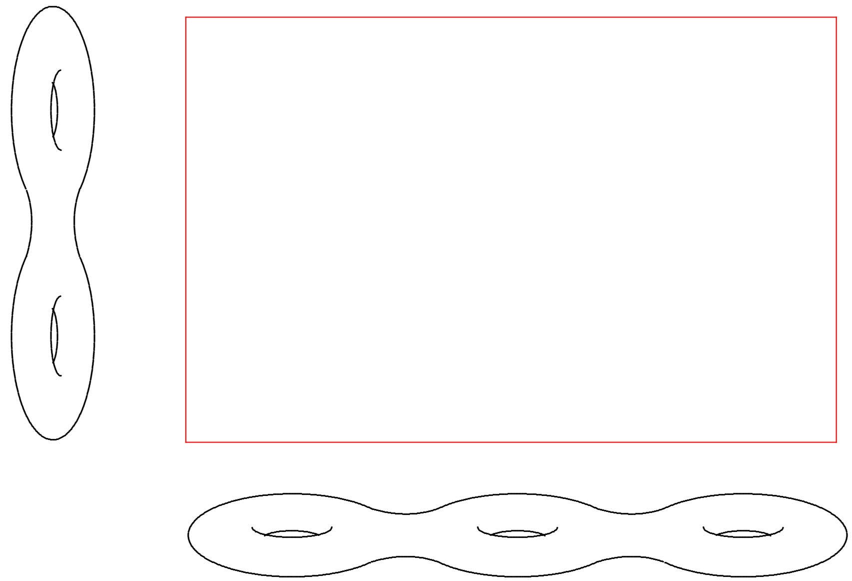

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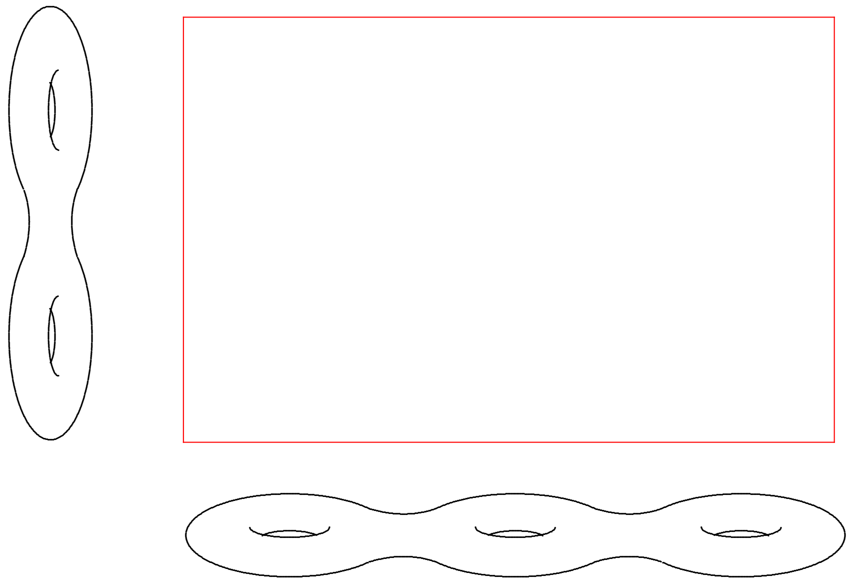

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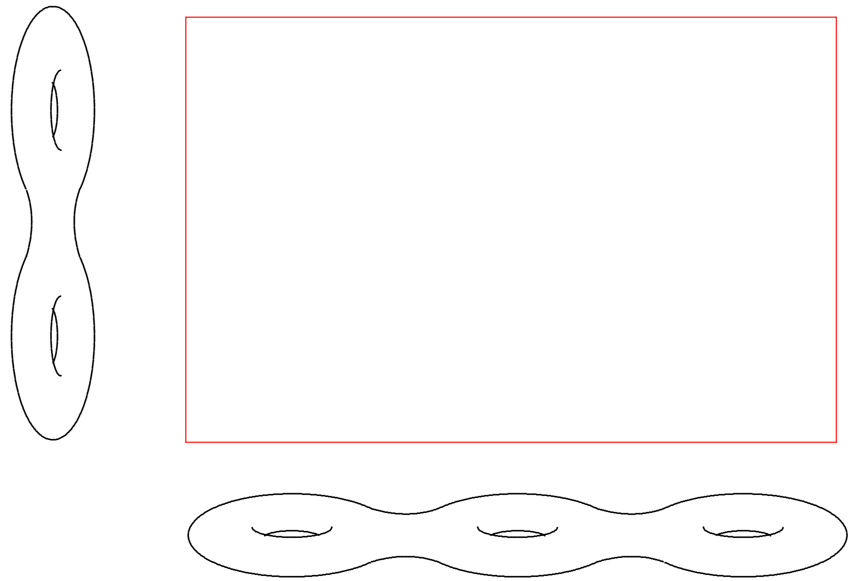

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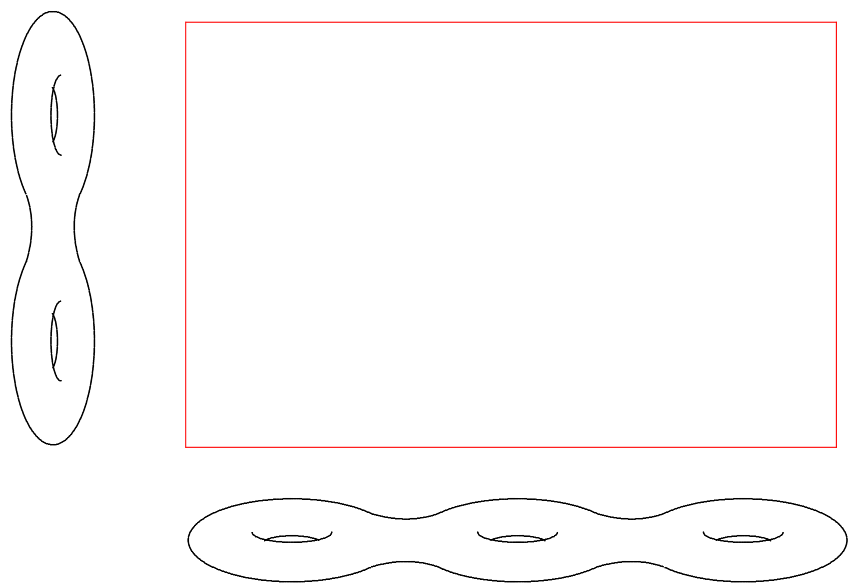

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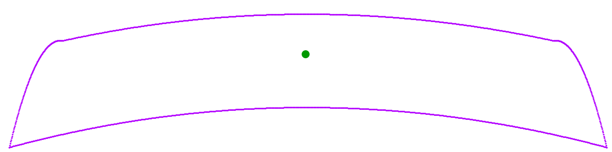

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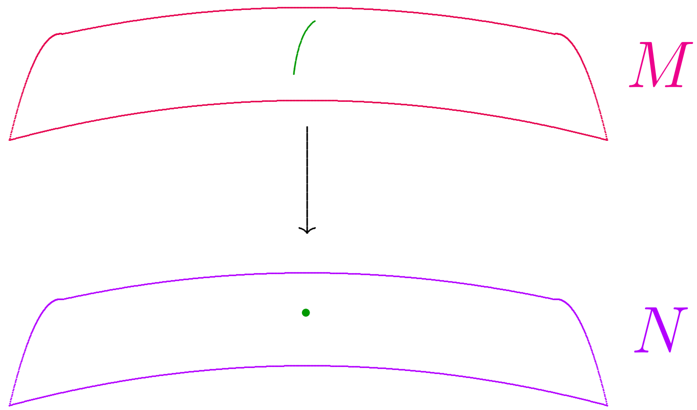

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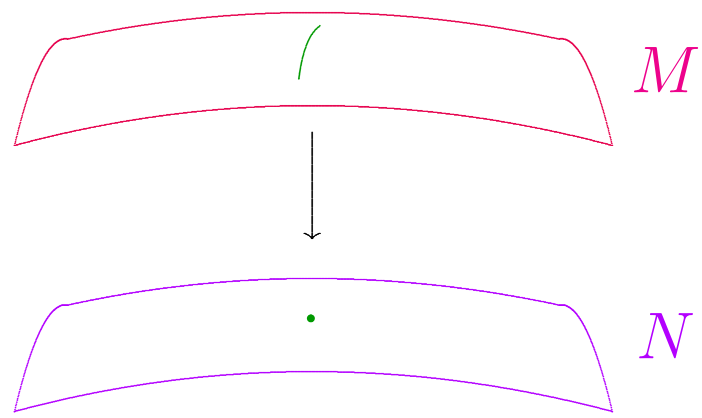

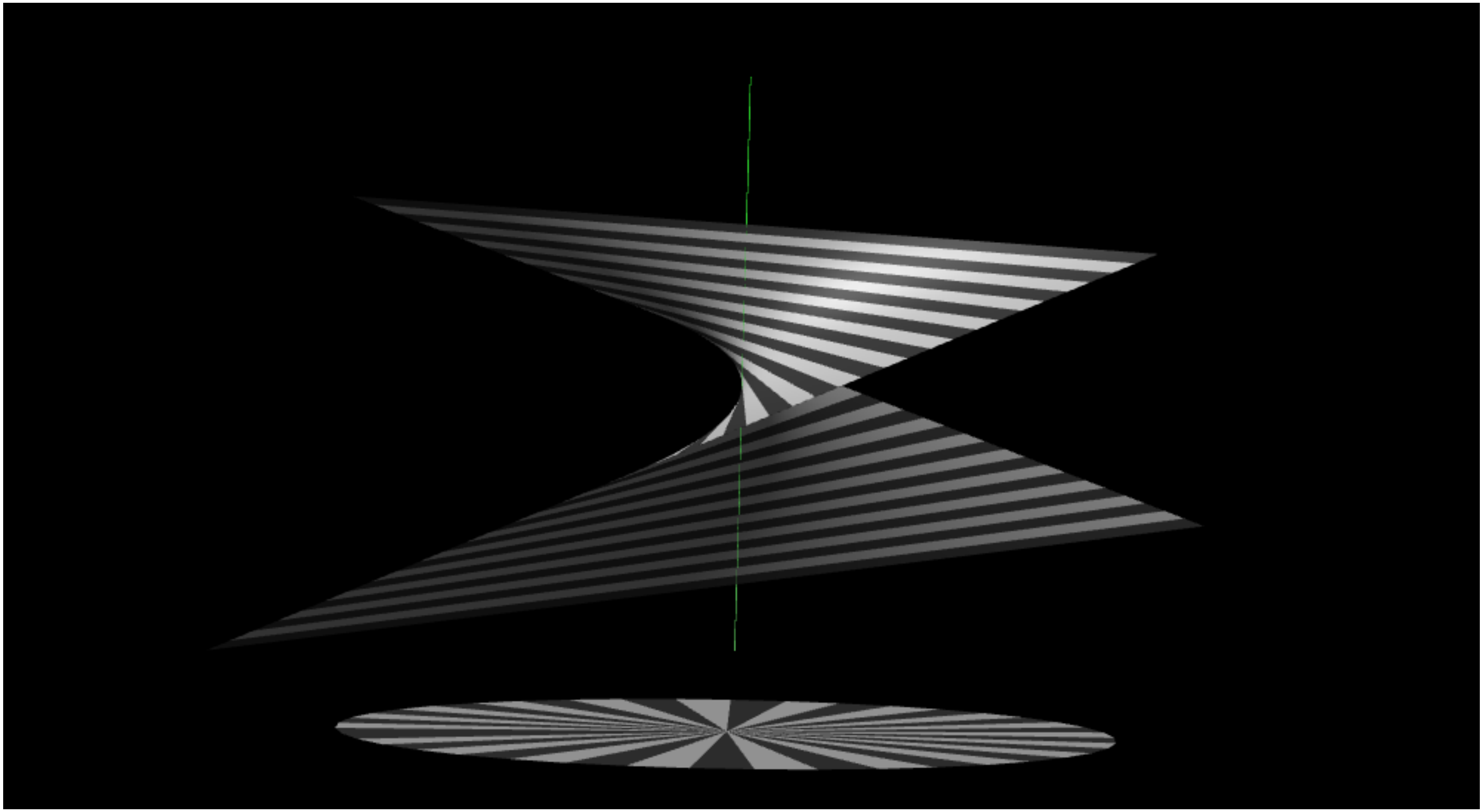

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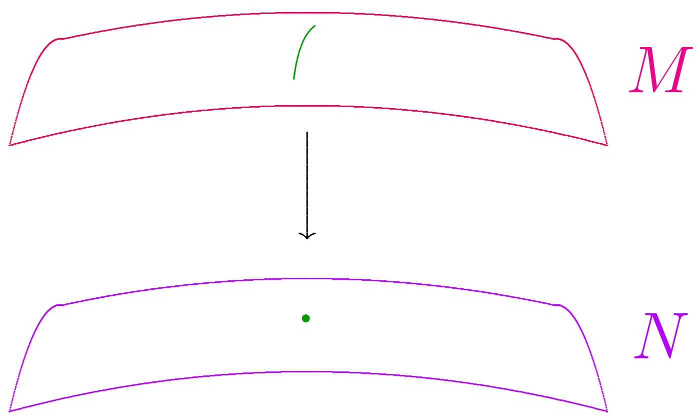

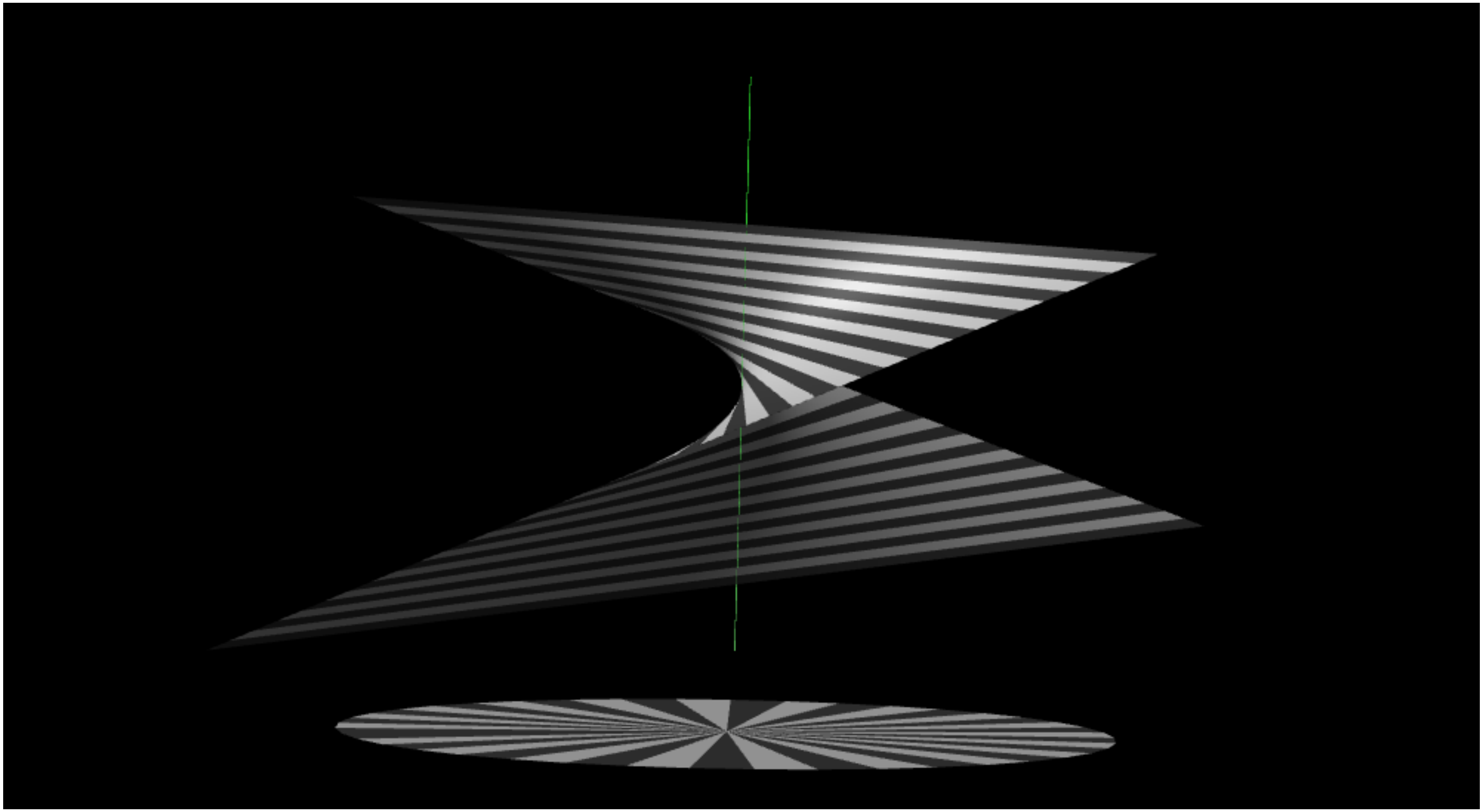

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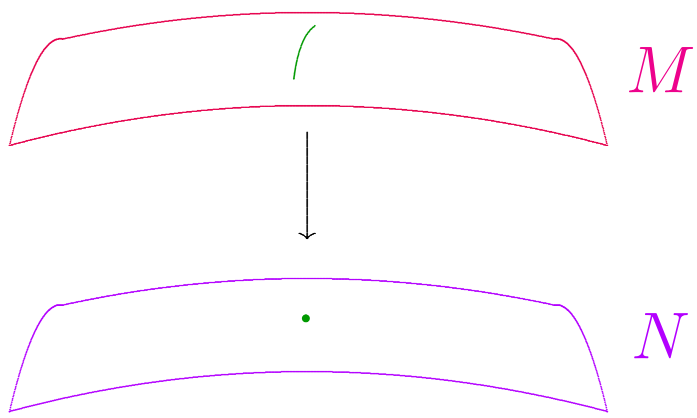

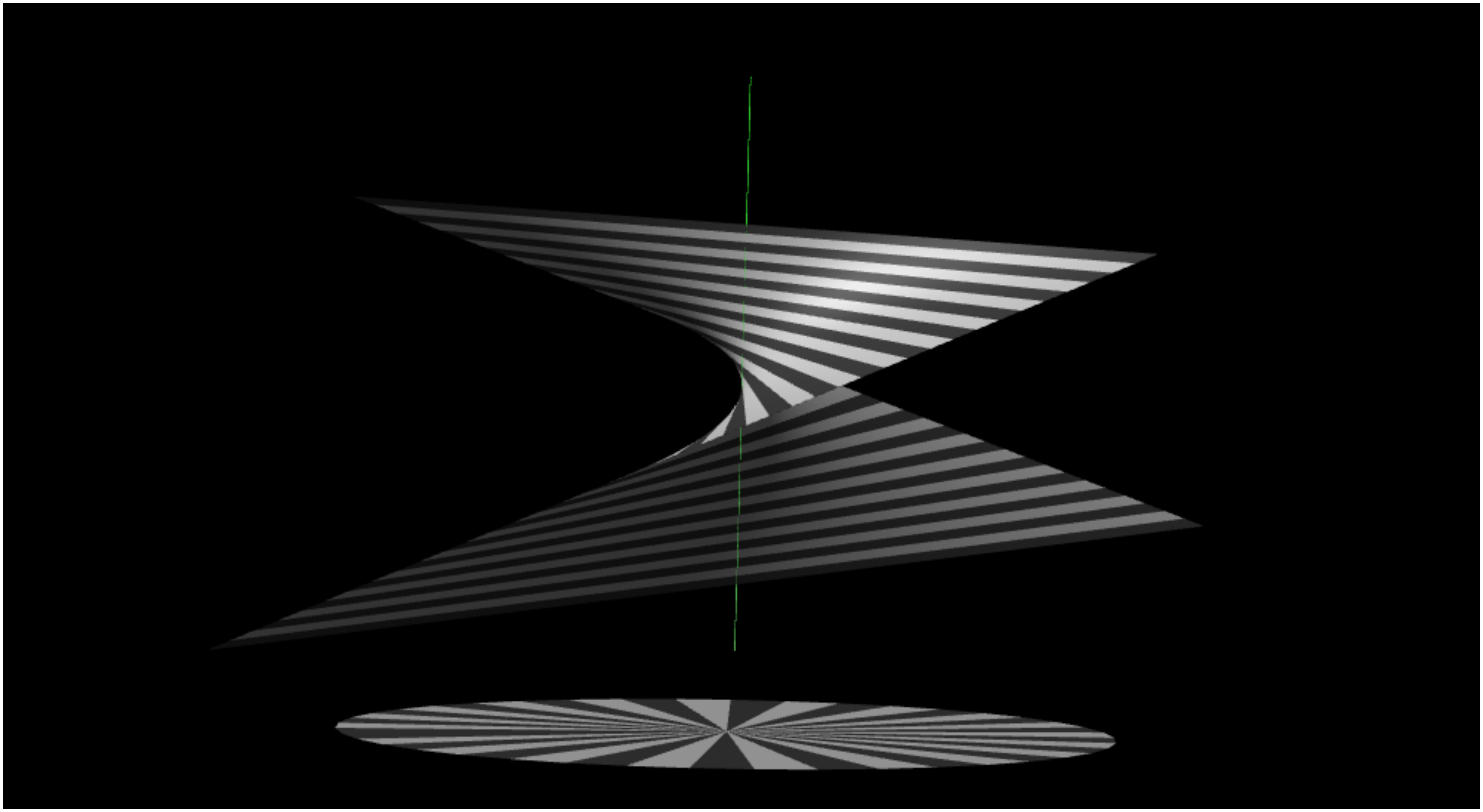

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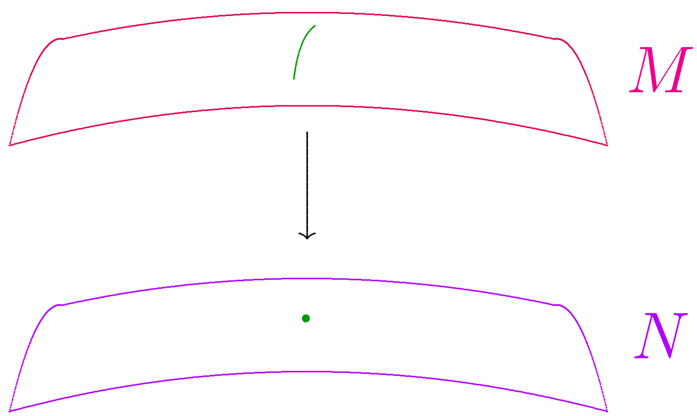

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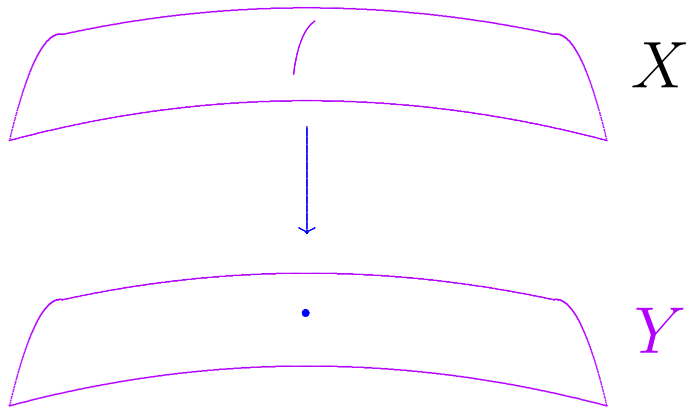

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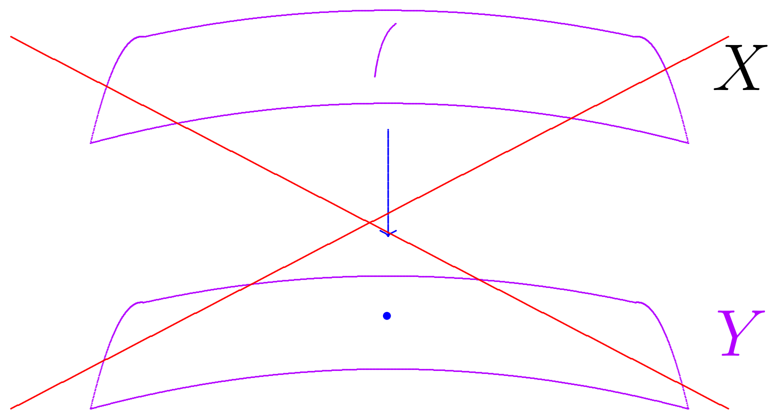

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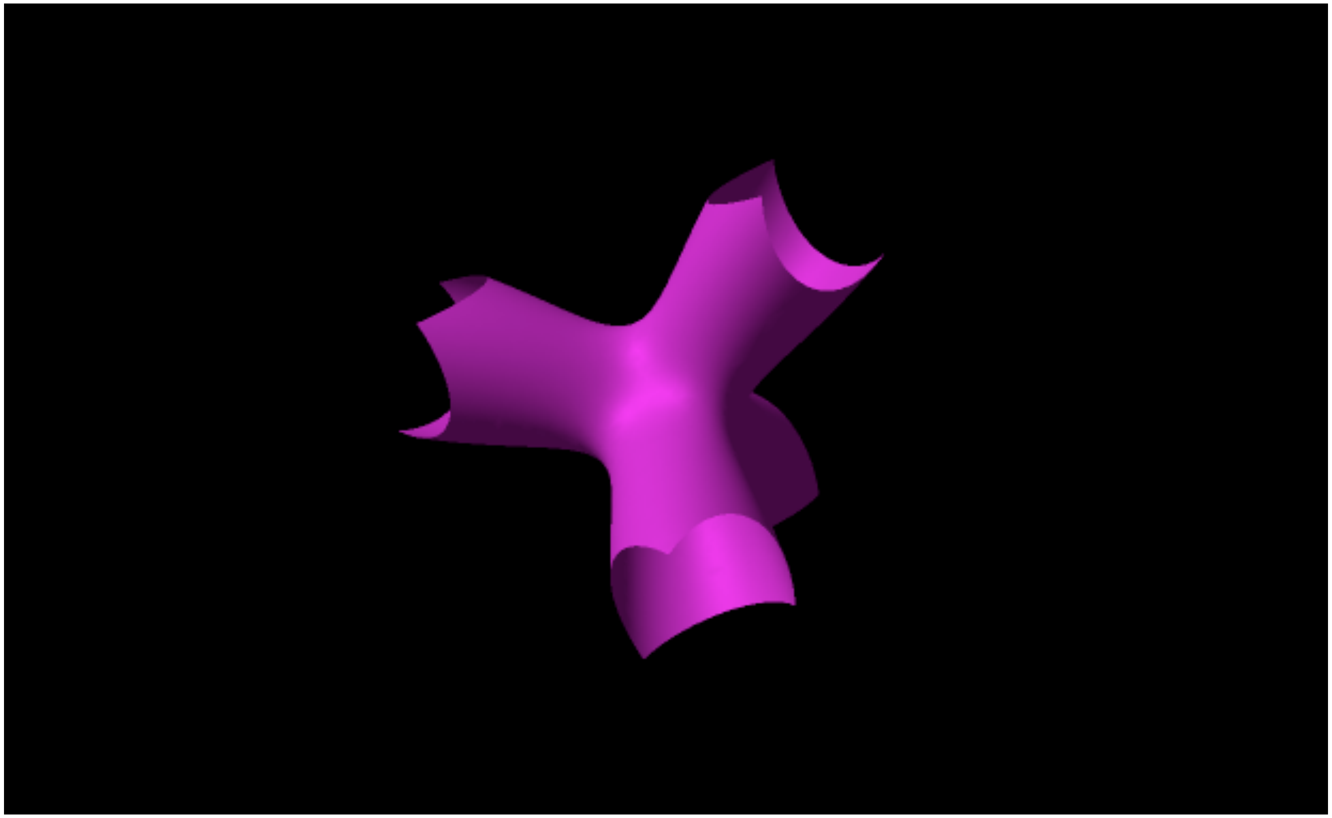

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≥ 5	"general type"	2	Yes

End of Lecture II