Cours de la chaire d'excellence

Fondation Sciences Mathématiques de Paris

Institut de Mathématiques
Jussieu - Paris Rive Gauche

Claude R. LeBrun

Stony Brook

Accueilli à l'IMJ-PRG (Sorbonne Univ., Univ. Paris Cité, CNRS)

Einstein Metrics, Four-Manifolds, and Differential Topology

Amphithéâtre Yvonne Choquet-Bruhat (bât. Perrin)

- Jeudi 19 mars 2026
- Jeudi 26 mars 2026
- Jeudi 2 avril 2026

Salle Pierre Grisvard (bât. Borel, 3e étage)

- Jeudi 9 avril 2026
- Jeudi 16 avril 2026
- Jeudi 23 avril 2026*

de 14h à 17h15

Institut Henri Poincaré

11, rue Pierre et Marie Curie, 75005 Paris

*La durée totale du cours étant de 15h, une séance parmi les trois dernières indiquées sera supprimée, en concertation avec Claude LeBrun.

Informations et résumé du cours

www.sciencesmaths-paris.fr

Einstein Metrics,

Four-Manifolds, &

Differential Topology I

Claude LeBrun
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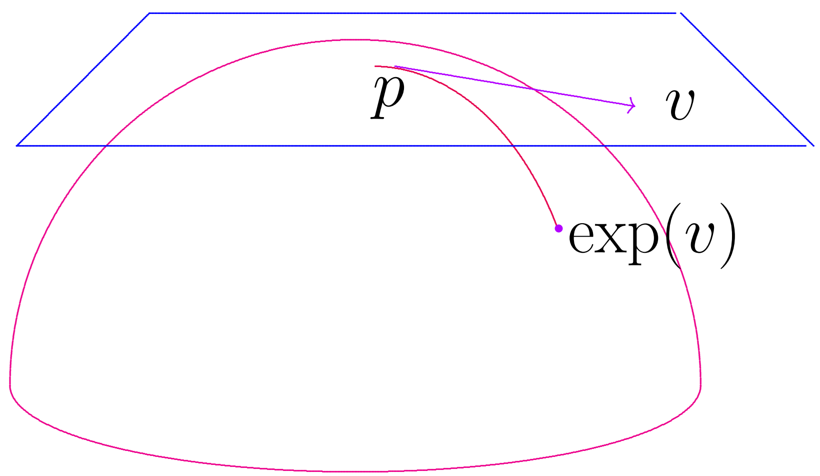

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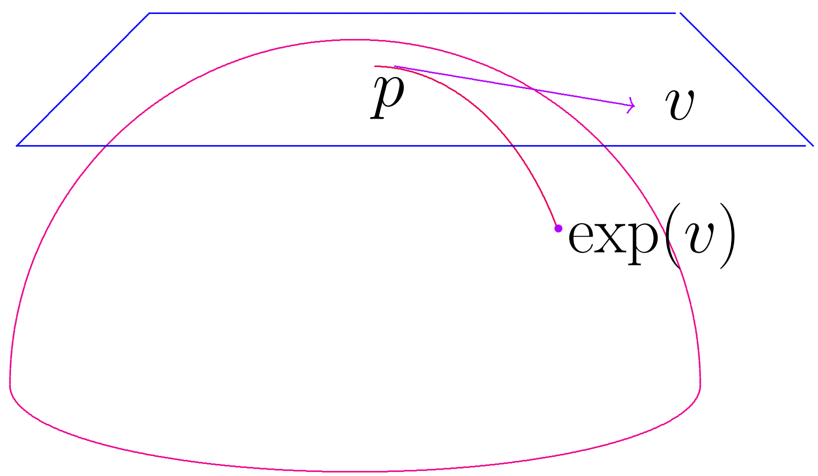

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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using the Einstein summation convention

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Components like $\mathcal{R}_{1212}$ are "sectional curvatures"...

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$$\left\{ \Pi \subset T_pM \text{ linear subspace } \middle| \; p \in M, \quad \Pi \cong \mathbb{R}^2 \right\}$$

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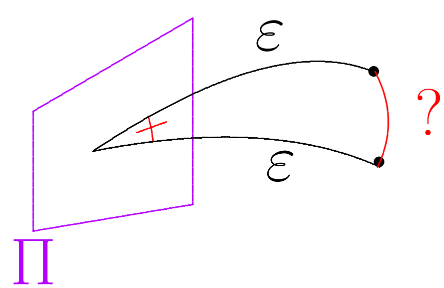

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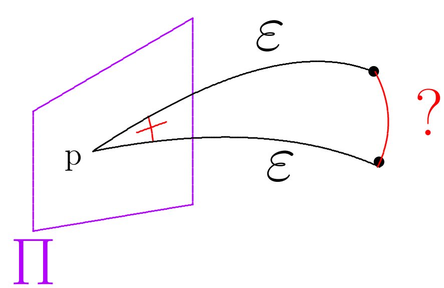

(assuming angle at p is sufficiently small)

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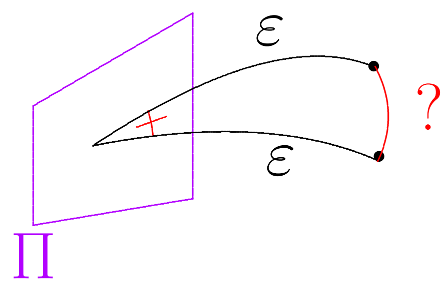

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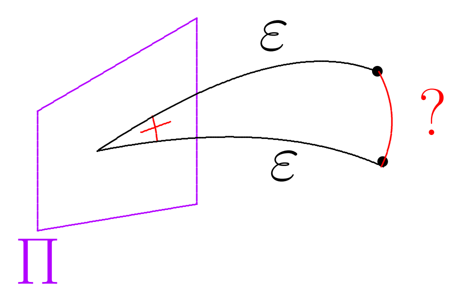

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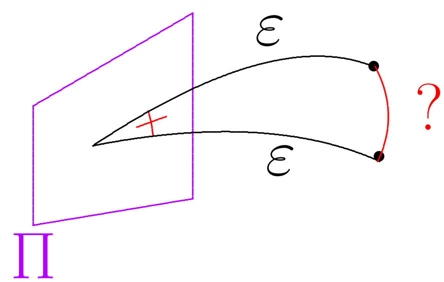

For example, $K \equiv +1$ for the unit n-sphere

$$S^n \subset \mathbb{R}^{n+1},$$

and this characterizes the standard metric on S^n.

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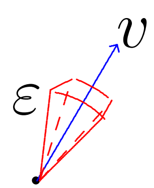

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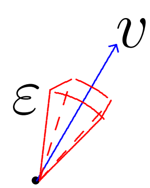

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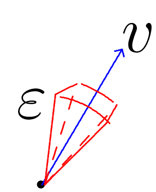

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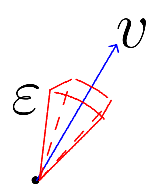

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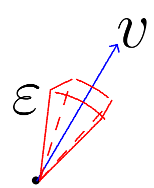

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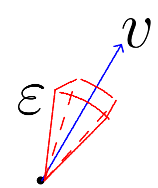

For example, the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$ has Ricci curvature $\equiv +(n-1)$,

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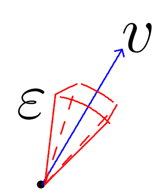

For example, the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$ has Ricci curvature $\equiv +(n-1)$, but this does not locally characterize the standard metric when $n \geq 4$.

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It's Einstein's notation for the gravitational field!

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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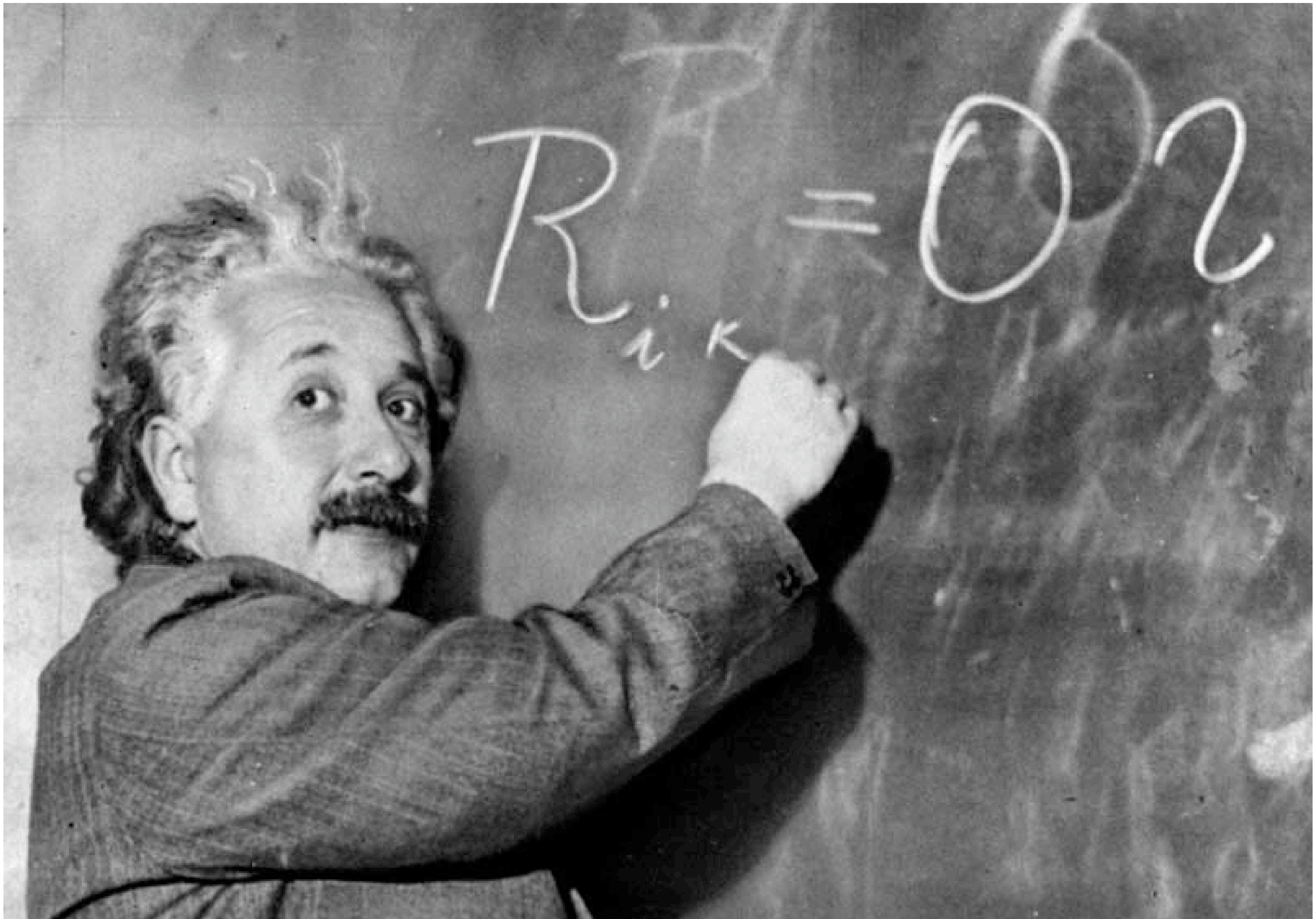

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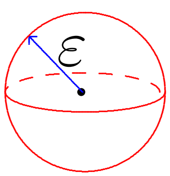

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$$\Delta = d^* d = - \operatorname{tr} \text{Hess}$$

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Proof.

$$\mathcal{R} \in \Gamma(\wedge^2 \otimes \mathrm{End}(TM))$$

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Proof. Contracted Bianchi identity

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In favorable circumstances, we will sometimes be able to use obstructions to show that these existence results are actually sharp!

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- If g is an Einstein metric on M, and $\Psi : M \to M$ is a diffeomorphism, then $\Psi^* g$ is also an Einstein metric. Thus, the space of solutions of the Einstein equation, when non-empty, is automatically infinite-dimensional. But (M, g) and $(M, \Psi^* g)$ are isometric, and so are indistinguishable from the point-of-view of Riemannian geometry.

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The $n = 4$ case is both unusual and fascinating.

Intermission

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature*

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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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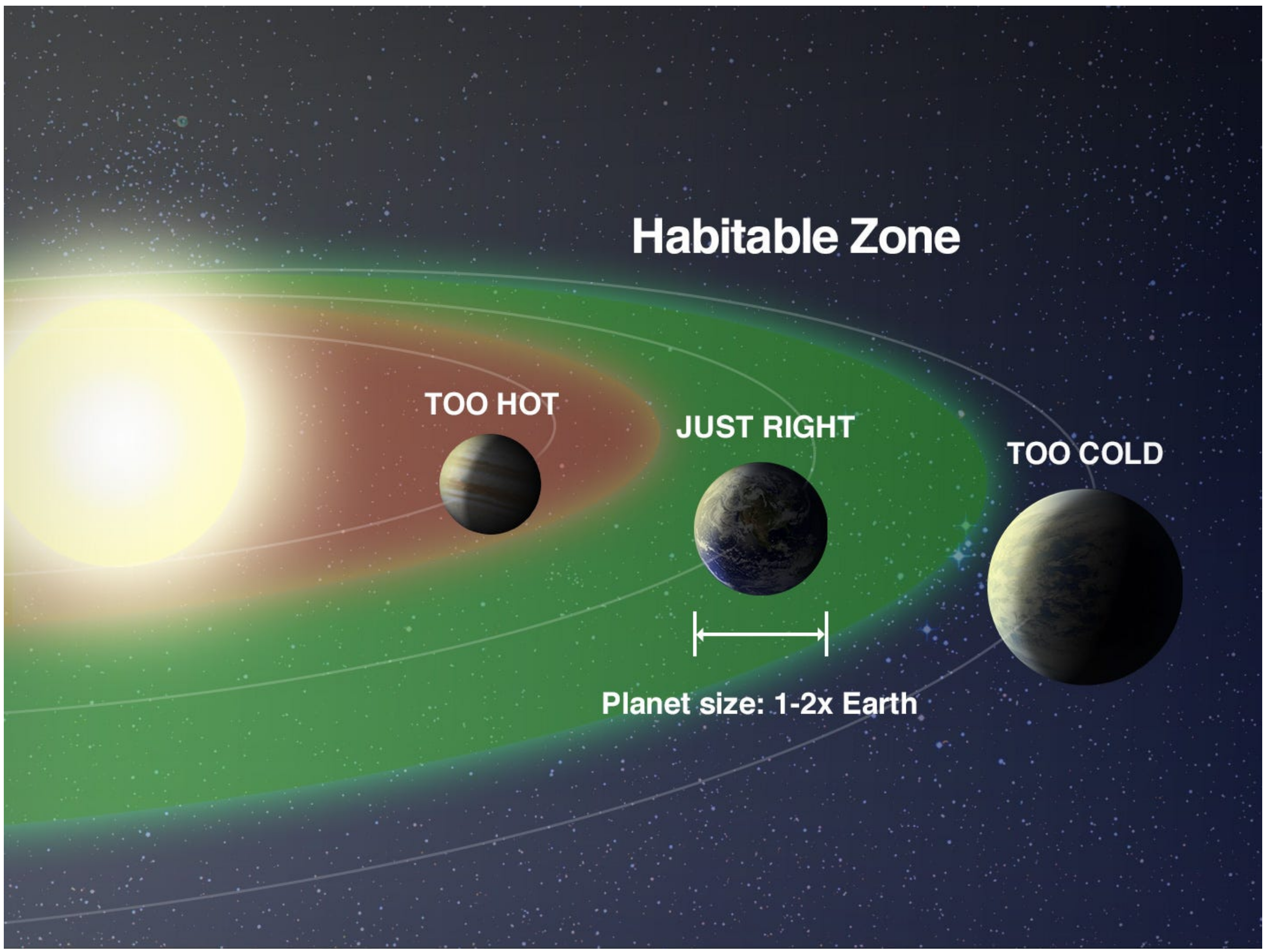

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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$K3 \approx$ generic quartic surface in $\mathbb{CP}_3$.

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$\{$skew 4×4 matrices$\}$ splits into two subspaces...

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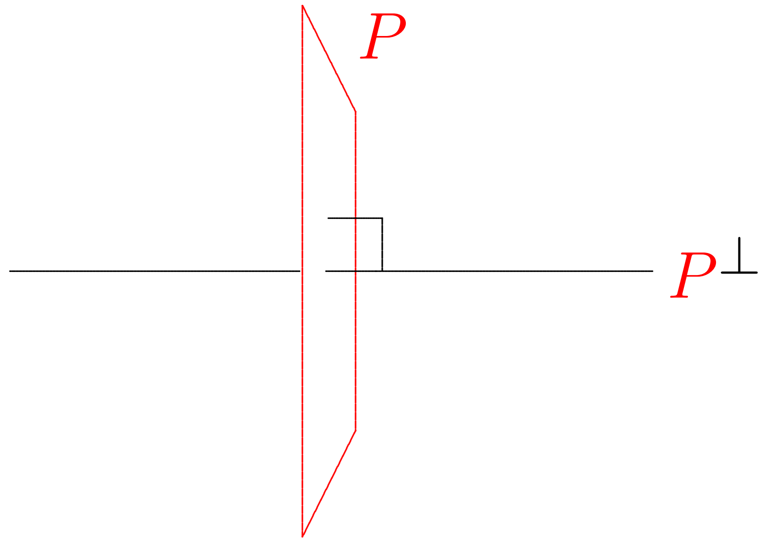

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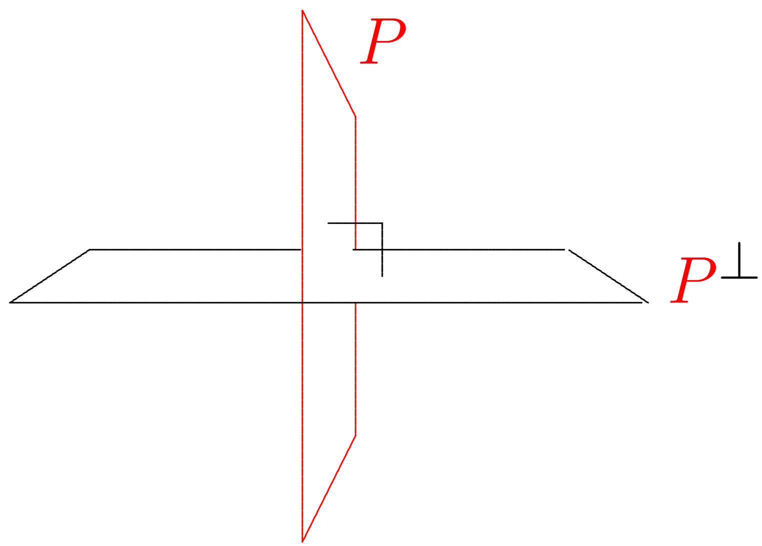

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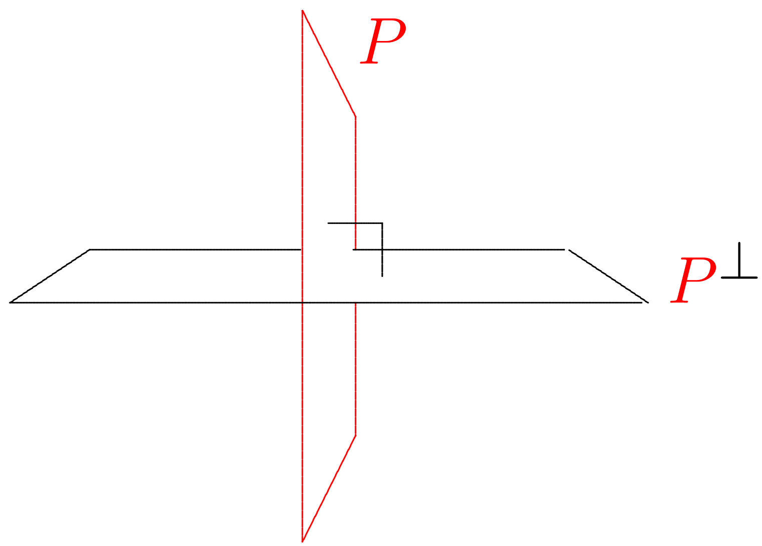

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We will further discuss the equality case later!

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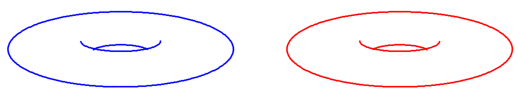

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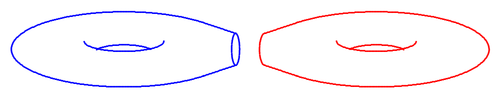

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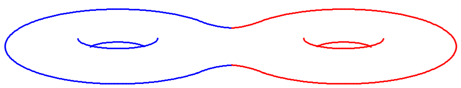

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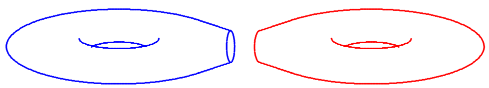

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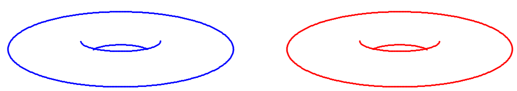

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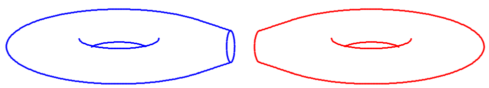

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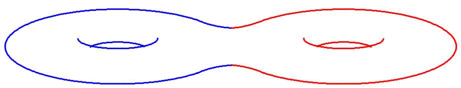

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Repeat for reversed orientation: also $4 + 5k > j$.

End of Lecture I