Mass in

Kähler Geometry

Claude LeBrun
Stony Brook University

Differential Geometry in the Large
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Joint work with
Joint work with

Hans-Joachim Hein
University of Maryland
Joint work with

Hans-Joachim Hein
Fordham University
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Definition. A complete, non-compact Riemannian $n$-manifold $(M^n, g)$ is called asymptotically Euclidean (AE) if there is a compact set $K \subset M$ such that each component of $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$. 
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\[\text{Diagram of a compact set with a compact manifold attached.}\]
**Definition.** A complete, non-compact Riemannian $n$-manifold $(M^n, g)$ is called asymptotically Euclidean (AE) if there is a compact set $K \subset M$ such that $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that

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m(M, g) := \left[ g_{ij,i} - g_{ii,j} \right]
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\[ m(M, g) := \lim_{\rho \to \infty} \Gamma \left( \frac{n^2}{4} (n - 1) \pi^{n/2} \right) \int \Sigma (g_{ij,i} - g_{ii,j}) \nu^j \]
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**Chruściel-type fall-off:**

$$g_{jk} - \delta_{jk} \in C^1_{-\tau}, \quad \tau > \frac{n - 2}{2}$$
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When \( n = 3 \), ADM mass in general relativity.
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$$g = - \left( 1 - \frac{2m}{\rho^{n-2}} \right) dt^2 + \left( 1 - \frac{2m}{\rho^{n-2}} \right)^{-1} d\rho^2 + \rho^2 h_{S^{n-1}}$$
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also has mass $m$. 

Positive Mass Conjecture:
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Physical intuition:
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Any AE manifold with \( s \geq 0 \) has \( m \geq 0 \).

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Local matter density \( \geq 0 \)
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Physical intuition:
Local matter density $\geq 0 \implies$ total mass $\geq 0$. 
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Proved in dimension $n \leq 7$.

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Proved for spin manifolds (implicitly, for any $n$).

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Conjectured true in ALE case, too.

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on line bundles \( L \to \mathbb{CP}^1 \) of Chern-class \( \leq -3 \).
Mass of ALE Kähler manifolds?
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\textbf{Lemma.}
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**Upshot:**
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**Upshot:**

Mass of an ALE Kähler manifold is unambiguous.
Mass of **ALE Kähler** manifolds?

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**Upshot:**

Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!
We begin with the scalar-flat Kähler case.
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**Theorem A.**
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**Theorem A.** *The mass of an ALE*
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**Theorem A.** *The mass of an ALE scalar-flat Kähler manifold is a topological invariant.*
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- the Kähler class \( [\omega] \in H^2(M) \) of the metric.

In fact, we will see that there is an explicit formula for the mass in terms of these data!
The explicit formula reproduces the mass in cases where it previously had been laboriously computed from the definition.
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**Theorem B.** There are infinitely many topological types of ALE scalar-flat Kähler surfaces that have zero mass, but are not Ricci-flat.
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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)
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**Lemma.** Let \((M, g)\) be any ALE manifold of real dimension \(n \geq 4\). Then the natural map
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is an isomorphism.

Here
\[
H^p_c(M) := \frac{\ker d : \mathcal{E}^p_c(M) \to \mathcal{E}^{p+1}_c(M)}{d\mathcal{E}^{p-1}_c(M)}
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Compactify \(M\) as \(\overline{M} = M \cup (S^{n-1}/\Gamma)\).
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Explicit formula depends on a topological fact:

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induced by the inclusion of compactly supported smooth forms into all forms.
We can now state our mass formula:

**Theorem C.**
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\[
m(M, g) = -\langle c_1, \omega \rangle_{m-1} + \frac{(m-1)!}{4(2m-1)\pi^{m-1}} \int_M s g d\mu_g
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**Theorem C.** Any ALE Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

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where

- \(s = \text{scalar curvature};\)
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- \(s =\) scalar curvature;
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- \([\omega] \in H^2(M)\) is Kähler class of \((g, J)\); and
- \(\langle , \rangle\) is pairing between \(H_c^2(M)\) and \(H^{2m-2}(M)\).
\[ m(M, g) = -\frac{\langle \spadesuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g \]
\[
\frac{4\pi^m(2m-1)}{(m-1)!} m(M, g) = -\frac{4\pi}{(m-1)!} \langle \spadesuit (c_1), [\omega]^{m-1} \rangle + \int_M s_g d\mu_g
\]
For a compact Kähler manifold $(M^{2m}, g, J)$,

$$\int_M s_g d\mu_g = \frac{4\pi}{(m - 1)!} \langle c_1, [\omega]^{m-1} \rangle$$
For a compact Kähler manifold \((M^{2m}, g, J)\),

\[ 0 = -\frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle + \int_M s_g d\mu_g \]
For an ALE Kähler manifold \((M^{2m}, g, J)\),

\[
m(M, g) = -\frac{\langle ♣(c_1), [\omega]^{m-1}\rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g
\]
For an ALE Kähler manifold \((M^{2m}, g, J)\),

\[
\frac{4\pi^m(2m-1)}{(m-1)!} m(M, g) = -\frac{4\pi}{(m-1)!} \langle \clubsuit (c_1), [\omega]^{m-1} \rangle + \int_M s_g d\mu_g
\]

So the mass is a “boundary correction” to the topological formula for the total scalar curvature.
Theorem C. Any ALE Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle \clubsuit (c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g
\]
Corollary. Any ALE scalar-flat Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle ♣(c_1), [\omega]^{m-1}\rangle}{(2m - 1)\pi^{m-1}}.
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Corollary. Any ALE scalar-flat Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle ♣(c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}}.
\]

So Theorem A is an immediate consequence!
Rough Idea of Proof:
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Special Case: Suppose
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(z^1, z^2) = (x^1 + ix^2, x^3 + ix^4)
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are harmonic. So \( x^j \) are harmonic, too, and

\[
g^{jk} \left( g_{j\ell,k} - g_{jk,\ell} \right) \nu^\ell \alpha_E = -\star d \log \left( \sqrt{\det g} \right) + O(\varrho^{-3-\varepsilon}).
\]
\[ m(M, g) = - \lim_{\rho \to \infty} \frac{1}{12\pi^2} \int_{S_\rho/\Gamma} \star d \left( \log \sqrt{\det g} \right) \]
\[ m(M, g) = - \lim_{e \to \infty} \frac{1}{12\pi^2} \int_{S_e/\Gamma} \star d \left( \log \sqrt{\det g} \right) \]

Now set \( \theta = \frac{i}{2}(\partial - \bar{\partial}) \left( \log \sqrt{\det g} \right) \), so that
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However, since \( s = 0 \),
\[ d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4} \omega^2 = 0. \]
\[ m(M, g) = - \lim_{\varrho \to \infty} \frac{1}{12\pi^2} \int_{S_{\varrho}/\Gamma} \ast d \left( \log \sqrt{\det g} \right) \]

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![Graph showing a smooth cut-off function](image-url)
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Compactly supported, because $d\theta = \rho$ near infinity.
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where $M_\rho$ defined by radius $\leq \rho$. 

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scalar-flat $\implies \rho \wedge \omega = 0$. 
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\[ [\psi] = \clubsuit([\rho]) = 2\pi \bullet (c_1) \in H_c^2(M) \]

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because scalar-flat \( \Rightarrow \rho \wedge \omega = 0. \)
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by Stokes’ theorem.
Let $f : M \to \mathbb{R}$ be smooth cut-off function:

<table>
<thead>
<tr>
<th>Equivalent</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\equiv 0$</td>
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m(M, g) = -\frac{1}{3\pi} \langle \ Sp\left(c_1\right), [\omega]\rangle
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as claimed.
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- $m = 2$;
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The last point is serious.
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Example: Honda metrics. Deform \( \mathcal{O}(-3) \to \mathbb{C}P_1 \).
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One argument proceeds by osculation:
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One argument proceeds by osculation:

\[ J = J_0 + O(\varrho^{-3}), \quad \nabla J = O(\varrho^{-4}) \]

in suitable asymptotic coordinates adapted to $g$. 
To understand $J$ at infinity:
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Let $\tilde{M}_\infty$ be universal over of end $M_\infty$. 
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Cap off $\tilde{M}_\infty$ by adding $\mathbb{CP}^{m-1}$ at infinity.
To understand $J$ at infinity:

Let $\widetilde{M}_\infty$ be universal over of end $M_\infty$.

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Added hypersurface $\mathbb{CP}^{m-1}$ has normal bundle $\mathcal{O}(1)$. 
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Belongs to $m$-dimensional family of hypersurfaces.
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Moduli space carries $\mathcal{O}$ projective structure
To understand $J$ at infinity:

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Added hypersurface $\mathbb{C}P^{m-1}$ has normal bundle $\mathcal{O}(1)$.

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Moduli space carries $\mathcal{O}$ projective structure

with many totally geodesic hypersurfaces.
To understand $J$ at infinity:

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So flat if $m \geq 3$. 
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When $m = 2$, Cotton tensor may be non-zero, but “flatter” than might naively expect.
To understand $J$ at infinity:
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**AE case:**

Compactify $M$ itself by adding $\mathbb{CP}^{m-1}$ at infinity.
To understand $J$ at infinity:

**AE case:**

Compactify $M$ itself by adding $\mathbb{CP}^{m-1}$ at infinity.

Linear system of $\mathbb{CP}^{m-1}$ gives holomorphic map

$$\overline{M} \to \mathbb{CP}^m$$

which is biholomorphism near $\mathbb{CP}^{m-1}$. 
To understand $J$ at infinity:

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This has some interesting consequences...
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Proof actually shows something stronger!
Theorem E (Penrose Inequality).

Let $(M,g,J)$ be an AE Kähler manifold with scalar curvature $s \geq 0$. Then $(M,J)$ carries a canonical divisor $D$ that is expressed as a sum $\sum j_n D_j$ of compact complex hypersurfaces with positive integer coefficients, with the property that $\bigcup j D_j \neq \emptyset$ whenever $(M,J) \neq \mathbb{C}^m$. In terms of this divisor, we then have...
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so the mass formula implies the claim.
\[ m(M, g) = -\frac{\langle \clubsuit(c_1), \omega \rangle_{m-1}}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g \]
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Before ending,
Before ending, let me now say two more things...
Herzlichen Glückwunsch
zum Geburtstag, Wolfgang!
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E grazie a tutti gli organizzatori di questo bellissimo convegno!
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