Vol. 63, No. 3 DUKE MATHEMATICAL JOURNAL © August 1991

ON COMPLETE QUATERNIONIC-KÄHLER MANIFOLDS

CLAUDE LEBRUN

1. Introduction. This article is concerned with the existence of complete Riemannian metrics of special holonomy on $\mathbb{R}^{4n}$. We therefore begin by recalling the basic notions and results concerning holonomy groups; cf. [5] [6] [23].

Let (M, g) be a connected Riemannian m-manifold, and let $x \in M$ be a chosen basepoint. The *holonomy group* of (M, g, x) is the subgroup of $\mathrm{End}(T_xM)$ consisting of those transformations induced by parallel transport around piecewise-smooth loops based at x; the *restricted holonomy group* is similarly defined, using only loops representing $1 \in \pi_1(M, x)$. The latter is automatically a connected Lie group and may be identified with a Lie subgroup of $\mathrm{SO}(m)$ by choosing an orthogonal frame for T_xM. Changing the basepoint and/or frame only changes this subgroup by conjugation.

Excluding Riemannian products and symmetric spaces, very few subgroups of $\mathrm{SO}(m)$ can be restricted holonomy groups, as was first pointed out by Berger [4]. In fact, the full list is as follows: $\mathrm{SO}(m)$, $\mathrm{U}(m/2)$, $\mathrm{SU}(m/2)$, $\mathrm{Sp}(m/4) \times \mathrm{Sp}(1)/\mathbb{Z}_2$ $(m \geqslant 8)$, $G_2 (m = 7)$ and $\mathrm{Spin}(7)$ $(m = 8)$. In all but the first two cases, the manifold must be Einstein and must moreover be Ricci-flat except in the case of $\mathrm{Sp}(m/4) \times \mathrm{Sp}(1)/\mathbb{Z}_2$, for which the scalar curvature is *never* zero. A manifold of the latter holonomy group therefore resembles a symmetric space to an uncomfortable degree, and it behooves one to ask whether there are many or few complete manifolds of this type. In the positive scalar curvature case, there are no known complete nonsymmetric examples, and such are even known not to exist [21] in dimension 8; moreover, the moduli space of such metrics on a fixed manifold is a discrete space [15]; cf. [25]. In this article it will be shown that, by contrast, the moduli space of complete metrics on $\mathbb{R}^{4n}$ with holonomy $\mathrm{Sp}(n) \times \mathrm{Sp}(1)/\mathbb{Z}_2$ is infinite-dimensional. (The scalar curvature of these Einstein metrics is, of course, negative.)

A Riemannian manifold (M, g) of dimension $4n$, $n \geqslant 2$, will be called *quaternionic-Kähler* if its holonomy is (up to conjugacy) a subgroup of $\mathrm{Sp}(n)\mathrm{Sp}(1) := \mathrm{Sp}(n) \times \mathrm{Sp}(1)/\mathbb{Z}_2$, but not a subgroup of $\mathrm{Sp}(n)$. Here $\mathrm{Sp}(n) := \mathrm{GL}(n, \mathbb{H}) \cap \mathrm{SO}(4n)$, where $\mathbb{H}$ denotes the quaternions, and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ is the subgroup of $\mathrm{SO}(4n)$ consisting of transformations of $\mathbb{R}^{4n} = \mathbb{H}^n$ of the form

$$\vec{v} \mapsto A\vec{v}\,q^{-1}$$

where $A \in \mathrm{Sp}(n)$ and $q \in S^3 \subset \mathbb{H}$. Such a manifold is never a Riemannian product

Received 12 June 1990.
Research supported in part by an NSF grant.

and so has holonomy $\mathrm{Sp}(n)\mathrm{Sp}(1)$ unless it is symmetric; in the latter case, the holonomy is a proper subgroup of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ unless the manifold is locally isometric to either $\mathbb{HP}_n = \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times S(1)$ or its noncompact dual $\mathbb{H}\mathscr{H}_n = \mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$.

While we are not interested here in symmetric spaces in their own right, one can nonetheless learn a great deal from an intelligent examination of $\mathbb{HP}_n$. Notice that this is not a complex manifold; indeed, it does not even admit an almost-complex structure! This may seem confusing insofar as the tangent space of $\mathbb{HP}_n$ would seem in some sense to be a quaternionic vector space. The answer to this riddle lies in the fact that $\mathbb{H}$ has nontrivial automorphisms as a division ring, exactly corresponding to the $\mathrm{Sp}(1)$ factor of $\mathrm{Sp}(n)\mathrm{Sp}(1)$; if you like, there is a bundle of division rings, locally modelled on $\mathbb{H}$, over $\mathbb{HP}_n$, and each tangent space is a vector space over the corresponding noncommutative field.

One can untangle this complicated situation by passing to a 2-sphere bundle over $\mathbb{HP}_n$, namely $\mathbb{CP}_{2n+1} \to \mathbb{HP}_n$, where the projection is given by the Hopf map. Not only is the pullback of $T\mathbb{HP}_n$ a complex vector bundle over $\mathbb{CP}_{2n+1}$, but $\mathbb{CP}_{2n+1}$ is itself a complex manifold! It was independently discovered by Salamon [24] and Bérard-Bergery [3] that this situation has an analogue for any quaternionic-Kähler manifold.

To see this let (M^{4n}, g) be a quaternionic-Kähler manifold and let $F \to M$ denote the principal $\mathrm{Sp}(n)\mathrm{Sp}(1)$-bundle generated by parallel transport of an arbitrary orthonormal frame. Then setting $Z := F/(\mathrm{Sp}(n)U(1))$ yields a 2-sphere bundle $\pi: Z \to M$, and each element y of Z corresponds to an orthogonal complex structure

$$J_y\colon T_xM \to T_xM, \qquad J_y^2 = -1, \qquad g(J_yv, J_yw) = g(v, w)$$

on M. Here, $x = \pi(y)$ and $v, w \in T_xM$. Let $D \subset TZ$ denote the horizontal subspace with respect to the Levi-Civita connection of g. Since $\pi_*: D_y \to T_xM$ is an isomorphism of real vector spaces, we can lift J_y to be an endomorphism $(\mathscr{J}_1)_y: D_y \to D_y$, $(\mathscr{J}_1)_y^2 = -1$, so that $D \subset TZ$ becomes a complex vector bundle, with $\mathscr{J}_1$ defined to be scalar multiplication by $\sqrt{-1}$. On the other hand the fibers of π are oriented metric 2-spheres and so may be considered as Riemann surfaces; thus the vertical tangent space $V = \ker \pi_*$ also carries an endomorphism $\mathscr{J}_2: V \to V$ with $(\mathscr{J}_2)^2 = -1$. We may thus define an almost-complex structure $\mathscr{J}$ on $TZ = D \oplus V$ by $\mathscr{J} = \mathscr{J}_1 \oplus \mathscr{J}_2$. Remarkably, this almost-complex structure is *automatically* integrable; i.e., Z has $\mathbb{C}$-valued charts such that $\mathscr{J}$ becomes identically equal to the usual almost-complex structure $T\mathbb{C}^{2n+1} \to T\mathbb{C}^{2n+1}$ defined by scalar multiplication by $i = \sqrt{-1}$. Moreover, the distribution $D \subset TZ$ becomes a *holomorphic* subbundle of the tangent bundle, and the projection $TZ \to TZ/D$ becomes a holomorphic line-bundle-valued 1-form $\Theta \in \Gamma(Z, \Omega^1(L))$, where $L := TZ/D$, which satisfies

$$\Theta \wedge (d\Theta)^{\wedge n} \neq 0.$$

Such a 1-form is called a *complex contact structure* and in particular gives an isomorphism $L^{\otimes(n+1)} = K^{-1}$, where $K = \Omega^{2n+1}$ is the canonical line bundle. Finally, the map $\sigma: Z \to Z$, given by $J_y \mapsto -J_y$ and corresponding to the antipodal map on each metric 2-sphere $\pi^{-1}(x)$, is an antiholomorphic involution ($\sigma^2 = 1$) without fixed points.

Our definition of a quaternionic-Kähler manifold explicitly excluded the 4-dimensional case $n = 1$. Indeed, since $SO(4) = Sp(1)Sp(1)$, nothing interesting can generally be said about Riemannian 4-manifolds with this holonomy group. But one could instead ask under what conditions the almost-complex manifold Z constructed above is a complex contact manifold. The answer, discovered by Richard Ward [26], is that one should require that g be Einstein with nonzero scalar curvature and that the conformal curvature W should be *self-dual*; i.e., $W = *W$, where $*$ is the Hodge star operator, here acting on a bundle-valued 2-form. (This development historically predated and motivated the work of Salamon and Bérard-Bergery and in turn built on Penrose's analysis [19] of the Ricci-flat case; cf. [1].) We shall therefore define a quaternionic-Kähler 4-manifold to be a half-conformally flat Einstein 4-manifold with nonzero scalar curvature. (Here an orientable Riemannian manifold is called *half-conformally flat* if there is an orientation with respect to which the conformal curvature W satisfies $W = *W$.)

The real power of the twistor space stems from the fact that the Salamon correspondence is *invertible* [16] [18] [2]. Namely, given a complex contact manifold (Z, Θ) of dimension $2n + 1$ together with an antiholomorphic involution $\sigma: Z \to Z$, let M be the set of genus-0 compact complex curves $C \subset Z$ which are invariant under σ, have normal bundle isomorphic to $[\mathcal{O}(1)]^{\oplus 2n}$ (where $\mathcal{O}(1)$ is the divisor of a point in $\mathbb{CP}_1$), and are transverse to the distribution $D = \ker \Theta$. In general, of course, this set is empty, but if it is not, it is a real-analytic 4-manifold. Moreover, it naturally carries a pseudo-Riemannian metric of holonomy $Sp(n - l, l)Sp(1)$ for some $0 \leqslant l \leqslant n$. Finally, if Z is the twistor space of a quaternionic-Kähler manifold M', then M' is naturally isometric to one connected component M. Conversely, the germ of the geometry at a point $x \in M$ determines the germ of Z along the corresponding curve C up to biholomorphism.

In this paper will exploit this invertibility to construct an infinite-dimensional space of deformations of $\mathbb{H}\mathscr{H}_n = Sp(n, 1)/Sp(n) \times Sp(1)$ through complete quaternionic-Kähler metrics. We do this by first recognizing the twistor space Z of $\mathbb{H}\mathscr{H}_n$ as an open set of $\mathbb{CP}_{2n+1}$. We then deform an open neighborhood $\tilde{Z}$ of the closure of this set in such a way as to preserve both the complex contact structure and the involution $\sigma: \tilde{Z} \to \tilde{Z}$ by covering $\tilde{Z}$ with three open sets U_1, U_2, U_3 such that $\sigma(U_1) = U_2$, $\sigma(U_3) = U_3$ and $U_1 \cap U_2 = \varnothing$; we generate our deformations by replacing the identity map on $U_1 \cap U_3$ with an arbitrary complex contact transformation, while on $U_2 \cap U_3$ replacing the identity with the same contact transformation conjugated by σ. For small deformations of this type, we are then able to produce a complete quaternionic-Kähler manifold as one connected component of the σ-invariant rational curves transverse to the contact distribution.

Acknowledgements. I would like to thank Simon Salamon, Robin Graham, Jack Lee, and Roger Penrose for some stimulating conversations, and both the Institute

for Advanced Study and the Max-Planck-Institut for their hospitality and support.

2. Preliminaries. We begin our work with a careful description of the twistor correspondence for the noncompact symmetric space $\mathbb{H}\mathscr{H}_n = \mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$, hereafter referred to as *quaternionic hyperbolic space.* If we define (right) quaternionic projective n-space by $\mathbb{HP}_n = (\mathbb{H}^{n+1} - \{0\})/\sim$, where

$$(q_1, q_2, \ldots, q_{n+1}) \sim (q_1 q, q_2 q, \ldots, q_{n+1} q)$$

for all $q \in \mathscr{H} - \{0\}$, we may notice that left multiplication by $\mathrm{Sp}(n, 1) := 0(4h, k) \cap \mathrm{GL}(n + 1, \mathbb{H})$ acts transitively on the subset

$$\sum_{\ell=1}^{n} \|q_\ell\|^2 < \|q_{n+1}\|^2$$

with isotropy subgroup $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. We may therefore identify $\mathbb{H}\mathscr{H}_n$ with

$$\mathbb{HP}_n^+ = \{[q_1, \ldots, q_{n+1}] \mid \sum_{\ell=1}^{k} \|q_\ell\|^2 < \|q_{n+1}\|^2\},$$

whereas the latter may be realized as the open ball $B^{4n} = \{(q_1, \ldots, q_n) \in \mathbb{H}^n \mid \sum_{\ell=1}^{n} \|q_\ell\|^2 < 1\}$ via the inclusion $\mathbb{H}^n \hookrightarrow \mathbb{HP}_n$: $(q_1, \ldots, q_n) \mapsto [q_1, \ldots, q_n, 1]$. The metric is uniquely determined by the requirement that it be $\mathrm{Sp}(n, 1)$ invariant since $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ acts irreducibly on the tangent space of $\mathbb{HP}_n^+$ at $[0, \ldots, 0, 1]$, namely by the canonical representation of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ on $\mathbb{H}^n$.

The naturality of the twistor correspondence allows one to lift the isometry group $\mathrm{Sp}(n, 1)$ of $\mathbb{H}\mathscr{H}_n$ to act on the twistor space Z by holomorphic maps; thus Z is just $\mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times U(1)$ equipped with an invariant complex structure.

Now by identifying $\mathbb{C}^{2n+2}$ with $\mathbb{H}^{n+1}$ via

$$(z_1, z_2, \ldots, z_{2n+1}, z_{2n+2}) \leftrightarrow (z_1 + jz_2, \ldots, z_{2n+1} + jz_{2n+2}),$$

we have an embedding

$$\mathrm{GL}(n + 1, \mathbb{H}) \hookrightarrow \mathrm{GL}(2n + 2, \mathbb{C})$$

given by left multiplication; thus $\mathrm{Sp}(n, 1)$ now acts on $\mathbb{CP}_{2n+1}$. Moreover, it acts *transitively* on

$$\mathbb{CP}_{2n+1}^+ = \{[z_1, z_2, \ldots, z_{2n+1}, z_{2n+2}] \mid \sum_{\ell=1}^{2n} |z_\ell|^2 < |z_{2n+1}|^2 + |z_{2n+2}|^2\}$$

with isotropy subgroup $\mathrm{Sp}(n) \times \mathrm{U}(1)$.

Moreover, if we identify the tangent space of $\mathbb{CP}^+_{2n+1}$ at $[0, 0, \ldots, 1, 0]$ with $\mathbb{H}^n \oplus \mathbb{C} = \mathbb{C}^{2n+1}$ via inhomogeneous coordinates

$$(z_1 + jz_2, \ldots, z_{2n-1} + jz_{2n}, \zeta) \mapsto [z_1, z_2, \ldots, z_{2n+1}, 1, \zeta],$$

we notice that the isotropy representation is

$$(\mathrm{Sp}(n) \times U(1)) \times (\mathbb{H}^n \oplus \mathbb{C}) \to \mathbb{H}^n \oplus \mathbb{C}$$

$$((A, \lambda), (\vec{v}, \zeta)) \mapsto (A\vec{v}\lambda^{-1}, \lambda^{-2}\zeta).$$

But this action is complex linear with respect to only two complex structures, namely the standard one on $T\mathbb{CP}_{2n+1}$ and its complex conjugate, since the complex span of a vector in $\mathbb{H}^n \oplus \mathbb{C}$ now coincides with the fixed-point set of its isotropy in $\mathrm{Sp}(n) \times U(1)$, and a complex structure commuting with this action is necessarily in the orthogonal group because $\mathrm{Sp}(n) \times U(1)$ acts transitively on the unit spheres of $\mathbb{H}^n$ and $\mathbb{C}$.

It follows that the twistor space Z of $\mathbb{H}\mathscr{H}_n$ may be biholomorphically identified with $\mathbb{CP}^+_{2n+1}$. Moreover, the twistor projection $\pi: \mathbb{CP}^+_{2n+1} \to \mathbb{HP}^+_n$ just becomes

$$[z_1, z_2, \ldots, z_{2n+1}, z_{2n+2}] \mapsto [z_1 + jz_2, \ldots, z_{2n+1} + jz_{2n+2}],$$

while the real structure

$$\sigma: Z \to Z$$

just becomes right multiplication $[\vec{v}] \mapsto [\vec{v}j]$ by j; explicitly, it is the map σ: $\mathbb{CP}_{2n+1} \to \mathbb{CP}_{2n+1}$ given by $[z_1, z_2, \ldots, z_{2n+1}, z_{2n+2}] \mapsto [-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2n+2}, \bar{z}_{2n+1}]$.

Notice that the structures discussed so far precisely coincide with the restrictions of those of $\mathbb{HP}_n$ to the open ball $\mathbb{HP}^+_n \subset \mathbb{HP}_n$. This amounts to the observation that $\mathbb{H}\mathscr{H}_n$ and $\mathbb{HP}_n$ are *paraconformally equivalent* in the terminology of Bailey and Eastwood [2].

It is the *complex contact form* of the twistor space that distinguishes between our two distinct metrics which are related in this fashion. Again, by naturality, we seek a contact structure on $Z = \mathbb{CP}^+_{2n+1}$ which is invariant under the action of $\mathrm{Sp}(n, 1)$. Such a form may be constructed as follows. Let ω denote the complex symplectic form

$$\omega = dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n} - dz_{2n+1} \wedge dz_{2n+2},$$

which may be rewritten as

$$\omega(\bar{u}, \bar{v}) = \langle \vec{u}, \vec{v}j \rangle$$

where $\langle\,,\rangle$ denotes the pseudo-Hermitian inner product

$$\left(\sum_{\ell=1}^{2n} dz_\ell \otimes d\bar{z}_\ell\right) - (dz_{2n+1} \otimes d\bar{z}_{2n+1} + dz_{2n+2} \otimes d\bar{z}_{2n+2})$$

and so is invariant under the action of

$$Sp(n, 1) = GL(n+1, \mathbb{H}) \cap U(2n, 2).$$

We then let $\Theta \in \Gamma(\mathbb{CP}_{2+1}, \Omega^1(2))$ denote the line-bundle-valued 1-form given by

$$\Theta(p_{*\vec{u}}(\vec{v})) := \omega(\vec{u}, \vec{v})$$

where $p\colon \mathbb{C}^{2n+2} - \{\vec{0}\} \to \mathbb{CP}_{2n+1}$ is the canonical projection $\vec{u} \mapsto [\vec{u}]$ and where $T_u^{1,0} \to \mathbb{C}^{2n+2}$ is identified with $\mathbb{C}^{2n+2}$ in the obvious manner. This defines a line-bundle-valued 1-form precisely because $\omega(\vec{u}, \vec{u})$ always vanishes, and it takes its values in the Chern-class-2 line bundle $\mathcal{O}(2)$ because

$$p_{*\vec{u}}(\vec{v}) = p_{*(\lambda\vec{u})}(\lambda\vec{v})$$

for all $\lambda \in \mathbb{C} - \{0\}$. The invariance of ω implies that this 1-form is also invariant under the action of $\mathrm{Sp}(n, 1)$. Since we have already noticed that the isotropy representation acts on $\mathbb{H}^k \times \mathbb{C}$ by

$$(\vec{v}, \zeta) \mapsto (A\vec{v}\lambda^{-1}, \lambda^{-2}\zeta),$$

there is only one invariant complex hyperplane in the tangent space of $\mathbb{CP}^+_{2n+1}$, and this must therefore coincide with the annihilator of the above 1-form Θ. We conclude that Θ is the complex contact form on Z associated with the symmetric space metric on $\mathbb{H}\mathscr{H}_n$.

To summarize, we have proved the following.

LEMMA. *The twistor space of $\mathbb{H}\mathscr{H}_n = \mathbb{HP}^+_n$ is given by*

$$\mathbb{CP}^+_{2n+1} = \{[z_1, \ldots, z_{2n+2}] \mid \sum_{\ell=1}^{2n} |z_\ell|^2 < \sum_{\ell=2n+1}^{2n+2} |z_\ell|^2\}.$$

The real structure is given by

$$\sigma[z_1, z_{2n+1}, \ldots, z_{2n+1}, z_{2n+2}] = [-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2n+2}, \bar{z}_{2n+1}],$$

and the contact form is given by

$$\Theta = \left[\sum_{\ell=1}^{n} (z_{2\ell-1}dz_{2\ell} - z_{2\ell}dz_{2\ell-1})\right] - (z_{2n+1}dz_{2n+2} - z_{2n+2}dz_{2n+1}).$$

3. Deforming the twistor space. Let $a \in \mathbb{R}^+$ be any positive real number and let $\tilde{B}$ denote the $4n$-ball

$$\|q_1\|^2 + \cdots + \|q_n\|^2 < (1 + a)\|q_{n+1}\|^2$$

in $\mathbb{HP}_n$; this is then an open neighborhood of the closure of $\mathbb{H}\mathscr{H}_n \subset \mathbb{HP}_n$, which in our model is given by the unit ball $B \subset \mathbb{H}^n \subset \mathbb{HP}_n$. Let $\tilde{Z}$ denote the inverse image of $\tilde{B}$ via the Hopf map:

$$\tilde{Z} = \{[z_1, \ldots, z_{2n+2}] \in \mathbb{CP}_{2n+1} \,|\, |z_1|^2 + \cdots + |z_n|^2 < (1 + a)(|z_{2n+1}|^2 + |z_{2n+2}|^2)\}.$$

For any $1 > \varepsilon > \delta > 0$, this is covered by the three open sets

$$U_1 = \tilde{Z} \cap \{|z_{2n+1}|^2 < \varepsilon |z_{2n+2}|^2\},$$

$$U_2 = \tilde{Z} \cap \{|z_{2n+2}|^2 < \varepsilon |z_{2n+1}|^2\},$$

and

$$U_3 = \tilde{Z} \cap \left\{\frac{1}{\delta}|z_{2n+2}|^2 > |z_{2n+1}|^2 > \delta |z_{2n+2}|^2\right\},$$

and this cover of course satisfies $\bar{U}_1 \cap \bar{U}_2 \cap \bar{U}_3 = \varnothing$. Moreover, the real structure σ becomes an antiholomorphic identification of U_1 with U_2, whereas it acts on U_3 as an antiholomorphic involution.

On an open neighborhood of $\overline{U_1 \cap U_3}$, let f be any holomorphic section of the contact line bundle $\mathcal{O}(2) = K^{-1/(n+1)}$; e.g., we may take

$$f(z_1, \ldots, z_{2n+1}) = F\left[\frac{z_1}{z_{2n+2}}, \ldots, \frac{z_{2n}}{z_{2n+2}}\right]\frac{z_{2n+2}^3}{z_{2n+1}}$$

where F is an arbitrary holomorphic function on the ball $|\zeta_1|^2 + \cdots + |\zeta_{2n}|^2 < (1 + 2a)(1 + 2\varepsilon)$. We then associate to such an f the unique holomorphic vector field V_f such that

(1) $\Theta(V_f) = f$ and
(2) $\mathscr{L}_{V_f}\Theta \propto \Theta$.

Here Θ again represents the holomorphic contact form on $\mathbb{CP}_{2n+1}$ given by

$$\Theta = \left[\sum_{\ell=1}^{n} (z_{2\ell-1}dz_{2\ell} - z_{2\ell}dz_{2\ell-1})\right] - (z_{n+1}dz_{n+2} - z_{2n+2}dz_{2n+1}).$$

In fact, if we trivialize $\mathcal{O}(2)$ over $U_1 \cap U_3$ by introducing the affine chart $z_{n+2} = 1$, the contact structure is represented by the 1-form

$$\mathscr{V} = dz_{2n+1} + \sum_{\ell=1}^{n} (z_{2\ell-1}dz_{2\ell} - z_{2\ell}dz_{2\ell-1}),$$

and V_f is given explicitly by the formula

$$V_f = \frac{1}{2}\sum_{\ell=1}^{n}\left[(z_{2\ell-1}f_{2n+1} - f_{2\ell})\frac{\partial}{\partial z_{2\ell-1}} + (z_{2\ell}f_{2n+1} + f_{2\ell-1})\frac{\partial}{\partial z_{2\ell}}\right]$$
$$+ \left(f + \frac{1}{2}\sum_{\ell=1}^{n}(f_{2\ell-1}z_{2\ell-1} - f_{2\ell}z_{2\ell})\right)\frac{\partial}{\partial z_{2n+1}}.$$

For $t \in \mathbb{R}$ sufficiently small, we can then define

$$\Phi_{tf} := \exp(t \operatorname{Re} V_f)\colon U_1 \cap U_3 \to \mathbb{CP}_{2n+1};$$

this is automatically a biholomorphism preserving the contact structure. There is also an analogous biholomorphism

$$\tilde{\Phi}_{tf}\colon \exp(t \operatorname{Re} \tilde{V}_f)\colon U_2 \cap U_3 \to \mathbb{CP}_{2n+1}$$

where $\tilde{V}_f$ is the holomorphic contact vector field induced by $\sigma_* f$; thus

$$\operatorname{Re} \tilde{V}_f = \sigma_* \operatorname{Re} V_f$$

and

$$\Phi_{tf}\sigma = \sigma\tilde{\Phi}_{tf}.$$

Notice that our notation is defined such that $\Phi_f\colon U_1 \cap U_3 \to \mathbb{CP}_{2n+1}$ is actually well defined for all f in a neighborhood of the origin in the space of holomorphic functions on any fixed open neighborhood of $\overline{U_1 \cap U_3}$ with respect to the uniform topology. By shrinking this neighborhood we may assume that $\Phi_f(U_1 \cap U_3) \cap \tilde{\Phi}_f(U_2 \cap U_3) = \varnothing$.

Given an f for which Φ_f is defined as above, we will now define a new complex contact manifold. Roughly speaking, we wish to glue U_1 to U_3 via Φ_f, to U_2 to U_3 via $\tilde{\Phi}_f$. To make this precise let us start by defining

$$\hat{Z}_f = (U_1 \textstyle\coprod U_2 \coprod U_3)/\sim$$

where

$$U_1 \ni x \sim y \in U_3 \Leftrightarrow x \in U_1 \cap U_3,\ y = \Phi_f(x)$$

and

$$U_2 \ni x \sim y \in U_3 \Leftrightarrow x \in U_2 \cap U_3,\ y = \tilde{\Phi}_f(x).$$

This fails to be a complex manifold only because it need not be Hausdorff.

We remedy this by the following procedure. Let Σ denote the real hypersurface $|z_{2n+1}|^2 = (\varepsilon + \delta)|z_{2n+2}|^2/2$ and let $\Psi\colon \mathscr{U} \to \mathbb{CP}_{2n+1}$ denote $\exp(\mathrm{Re}(V_f))$, where $\mathscr{U}$ is some open set containing $\overline{U_1 \cap U_2}$. Let $\tilde{\Sigma} = \Psi^{-1}(\Sigma)$, which will be closed and connected in $\tilde{B}$ provided that f is assumed to be small. Then $\tilde{\Sigma}$ divides $\tilde{Z}$ into two regions since it is an oriented, closed, connected hypersurface in an orientable manifold; assuming that f is sufficiently small, one of these regions, which we will call $\tilde{U}_1$, is contained in U_1. We similarly define $\tilde{U}_2$, so that $\sigma(\tilde{U}_1) = \tilde{U}_2$. Finally, let $\tilde{U}_3$ denote the subset of $\tilde{Z}$ given by

$$\frac{2}{\varepsilon + \delta}|z_{2n+2}|^2 > |z_{2n+1}|^2 > \frac{\varepsilon + \delta}{2}|z_{2n+2}|^2.$$

Then the image of $\tilde{\tilde{U}}_1 \coprod \tilde{\tilde{U}}_2 \coprod \tilde{\tilde{U}}_3$ in $\hat{Z}_f$ is a topological manifold with boundary, and its interior $\tilde{Z}_f$ is the (Hausdorff) complex manifold which we will call the "deformation of $\tilde{Z}$ associated with f", provided that f is, of course, sufficiently small.

We now remark that $\tilde{Z}_f$ comes equipped with a complex contact structure and an antiholomorphic involution $\sigma_f\colon \tilde{Z}_f \to \tilde{Z}_f$. The former is just obtained by restricting the contact structure from $\mathbb{CP}_{2n+1}$ to U_1, U_2, and U_3, and remembering that our transition functions Φ_f and $\tilde{\Phi}_f$ preserve the annihilator distribution $D = \Theta^\perp \subset T\mathbb{CP}_{2n+1}$ of Θ, so that these induced structures on U_1, U_2, and U_3 agree on overlaps. The involution σ_f is defined to be $\sigma\colon U_1 \to U_2$ on U_1, $\sigma\colon U_2 \to U_1$ on U_2, and $\sigma\colon U_3 \to U_3$ on U_3; since $\sigma\Phi_f = \tilde{\Phi}_f\sigma$, this defines a consistent map on all of $\tilde{Z}_f$. In the next section, we will use these structures to create a quaternionic-Kähler manifold associated with $\tilde{Z}_f$.

4. The associated 4*n*-manifolds. For each sufficiently small holomorphic section f of the contact line bundle $\mathcal{O}(2) = K^{-1/(n+1)}$ on any fixed region $\mathscr{U} \supset U_1 \cap U_3$ in $\mathbb{CP}_{2n+1}$, we have produced a complex manifold $\tilde{Z}_f$. Moreover, if we consider the family $\tilde{Z}_{tf}$, $t \in (-\tilde{\varepsilon}, 1 + \tilde{\varepsilon})$, we obtain a real-analytic family of complex manifolds which displays $\tilde{Z}_f$ as a deformation of $\tilde{Z} \subset \mathbb{CP}_{2n+1}$. Moreover, each element of the family comes equipped with a real structure $\sigma_{tf}\colon \tilde{Z}_{tf} \to \tilde{Z}_{tf}$ and a complex contact structure

$$\Theta_{tf} \in \Gamma(\tilde{Z}_{tf}, \Omega^1(K^{-1/n+1})).$$

Now $\tilde{Z} \subset \mathbb{CP}_{2n+12}$ is foliated by complex projective lines $\mathbb{CP}_1$ which are invariant under $\sigma\colon \tilde{Z} \to \tilde{Z}$; moreover, the family of *all* complex projective lines forms a complex manifold of dimension $4n$. Because the normal bundle N of such a line is given by $[\mathcal{O}(1)]^{\oplus 2n} \to \mathbb{CP}_1$ and so satisfies $H^1(\mathbb{CP}_1, N) = 0$, Kodaira's stability theorem [12] implies that there is a complete $4n$-dimensional complex family of compact complex curves in $\tilde{Z}_{tf}$ for t small. (Kodaira's theorem is stated for complex analytic families, but our real-analytic family can easily be extended to a holomorphic one by analytic continuation.) Moreover, the given curves are deformations of the complex projective line and so are themselves $\mathbb{CP}_1$'s. Finally, the normal bundles of these curves are deformations of $N = [\mathcal{O}(1)]^{\oplus 2n}$; since $H^1(\mathbb{CP}_1, N \otimes N^*) = 0$, an open set (in the analytic Zariski topology) of these curves has normal bundle $N = [\mathcal{O}(1)]^{\oplus 2n}$. Since N is generated by its sections, these curves fill out an open set in each $\tilde{Z}_{tf}$.

Let $\mathcal{M}_{tf}$ denote the family of all compact genus zero curves C_x in $\tilde{Z}_{tf}$ of normal bundle $N = [\mathcal{O}(1)]^{\oplus 4n}$. On each $\tilde{Z}_{tf}$ we have a real structure $\sigma_{tf}\colon \tilde{Z}_{tf} \to \tilde{Z}_{tf}$, and this induces an antiholomorphic involution $\rho_{tf}\colon \mathcal{M}_{tf} \to \mathcal{M}_{tf}$, sending a rational curve $C \subset \tilde{Z}_{tf}$ to $\sigma_{tf}(C)$. Let $\tilde{M}_{tf}$ denote the fixed-point set of p_{tf}. By Kodaira's theorem [12] (see also [7], [22]), $\mathcal{M}_{tf}$ is a nonempty, complex $4n$-manifold for small t, and $\tilde{M}_{tf}$ is a real-analytic $4n$-manifold which sits in $\mathcal{M}_{tf}$ as a real slice. Note that $\tilde{M}_{tf}$ is nonempty for small t because it is not empty when $t = 0$.

The union of the $\tilde{M}_{tf}$, $t \in \mathbb{R}$, naturally forms a real-analytic $(4n + 1)$-manifold $\tilde{M}_{\mathbb{R}f}$, and $\tilde{M}_{\mathbb{R}f}$ comes equipped with a real-analytic submersion $\tilde{M}_{\mathbb{R}f} \xrightarrow{t} \mathbb{R}$ onto some interval about 0. Since $\mathbb{H}\mathcal{H}_n \subset \tilde{M}_0$ has a precompact neighborhood diffeomorphic to $\mathbb{R}^{4n}$, there is an open neighborhood $\hat{M}_{\mathbb{R}f}$ of $\mathbb{H}\mathcal{H}_n$ in $\tilde{M}_{\mathbb{R}f}$ which is diffeomorphic to $\mathbb{R}^{4n} \times I$, where I is some open interval about $0 \in \mathbb{R}$ in such a manner that the projection $\mathbb{R}^{4n} \times I \to I$ is just the function t.

We will henceforth refer to any rational curve C in a complex $(2n + 1)$-manifold Z as a *twistor line* if it has normal bundle $N = [\mathcal{O}(1)]^{\oplus 2n}$. If Z is equipped with a real structure $\sigma\colon Z \to Z$, we will call a σ-invariant twistor line a *real* twistor line. For instance we have defined $\tilde{M}_{tf}$ in the above discussion as an open subset of the real twistor lines in $\tilde{Z}_{tf}$. The twistor line corresponding to $x \in \mathcal{M}_{tf}$ will be denoted by $C_x \subset \tilde{Z}_{tf}$.

We will now need a technical lemma.

PROPOSITION 1. *Let (Z, Θ) be a complex contact manifold of dimension $2n + 1$, and assume that the space $\mathcal{M}$ of twistor lines in Z is nonempty. Then the set $\mathcal{S}$ of twistor lines tangent to the contact distribution $D = \Theta^\perp$ is a (possibly empty) nonsingular, closed complex hypersurface in $\mathcal{M}$.*

Proof. There is a double fibration

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Z & & \mathcal{M} \end{tikzcd}$$

relating Z and $\mathcal{M}$. Here $p\colon \mathcal{C} \to \mathcal{M}$ is a $\mathbb{CP}_1$-bundle, while the map $q\colon \mathcal{C} \to Z$ is a holomorphic submersion onto an open subset of Z and is injective on every fiber of p. (Indeed, $\mathcal{C} = \{(x, y) \in \mathcal{M} \oplus Z \mid y \in C_x\}$.) Now the contact form Θ is a 1-form with values in some $(n+1)$-st root $K^{-1/(n+1)}$ of the anticanonical bundle. But the restriction of K^{-1} to any twistor line is isomorphic (by the adjunction formula) to

$$T\mathbb{P}_1 \otimes \Lambda^{2n} N \cong \mathcal{O}(2) \otimes [\mathcal{O}(1)]^{\otimes 2n} = \mathcal{O}(2n+2),$$

so that the restriction of $K^{-1/(n+1)}$ to any twistor line must be isomorphic to $\mathcal{O}(2)$. Let Ω_p^1 denote $\Omega_c^1/p^*\Omega_{\mathcal{M}}^1$, i.e., the vertical cotangent bundle of p. Then the restriction of $q^*\Theta$ to the fibers of p is a section of $\Omega_p^1 \otimes q^*K^{-1/(n+1)}$ over $\mathscr{C}$. But on each fiber of p, $\Omega_p^1 \otimes q^*K^{-1(n+1)} \cong \mathcal{O}$; thus $\mathscr{L} := p_*^0(\Omega_p^1 \otimes q^*K^{-1/(n+1)})$ is a holomorphic line bundle, and $q^*\Theta$ pushes down as a section

$$\hat{\Theta} \in \Gamma(\mathcal{M}, \mathscr{L}).$$

Notice that $\mathscr{S}$ is by definition the zero locus of $\hat{\Theta}$.

To prove the proposition it therefore suffices to show that $d\hat{\Theta} \neq 0$ when $\hat{\Theta} = 0$. To do this let C_x be a twistor line tangent to $D = \Theta^\perp$; let $N_0 \subset N$ be the image of D in the normal bundle

$$N_0 := D/TC_x.$$

Let $u \in \Gamma(C_x, N)$ be a holomorphic section of the normal bundle which vanishes at some $y \in C_x$ but which has the property that $u'(y) \in N_y \otimes T_y^*C$ is not in $(N_0) \otimes T_y^*C$; this is possible because $[\mathcal{O}(1)]^{\oplus 2n}$ is very ample. Let $f\colon \mathbb{P}_1 \times \mathscr{V} \to Z$ be a family of curves with $f[\mathbb{P}_1 \times \{0\}] = C_x$, such that $[\partial f/\partial\zeta | \zeta = 0] = u$, where $\mathscr{V}$ is a small disk around $0 \in \mathbb{C}$ and the variable ζ is used for elements of $\mathscr{V}$. (This is possible because the family of twistor lines is *complete* in the sense of Kodaira; u just corresponds to an element of the tangent space $T_x\mathcal{M}$.) Choose any local coordinate η on $\mathbb{P}_1$ so that the point $y \in Z$ corresponds to $(\zeta, \eta) = (0, 0)$ and choose any local trivialization near y of $K^{-1/(n+1)}$, so that Θ is represented by a holomorphic 1-form ϑ. Then

$$\begin{aligned}\frac{\partial}{\partial\zeta}\left\langle f^*\vartheta, \frac{\partial}{\partial\eta}\right\rangle &= d(f^*\vartheta)\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left\langle f^*\vartheta, \frac{\partial}{\partial\zeta}\right\rangle \\ &= d\vartheta\left(f_*\frac{\partial}{\partial\zeta}, f_*\frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left\langle \vartheta, f_*\frac{\partial}{\partial\zeta}\right\rangle.\end{aligned}$$

But by construction $u = [f_*\partial/\partial\zeta]$ vanishes at y, whereas $\partial/\partial\eta\, \langle\vartheta, u\rangle$ does not. Thus the left-hand side, which represents the derivative of $\hat{\Theta}$ at our chosen point x of $\hat{\Theta} = 0$ in the direction of $u \in \Gamma(C_x, \mathcal{O}(N)) = T_x\mathcal{M}$, is nonzero. Q.E.D.

COROLLARY. *Let M denote the set of real twistor lines in a complex contact $(2n+1)$-manifold Z with antiholomorphic involution $\sigma\colon Z \to Z$ preserving the real structure. Then the set S of real twistor lines tangent to the contact distribution $D = \Theta^\perp$ is a smooth closed hypersurface.*

Proof. S is a real slice of the previously analyzed set $\mathscr{S}$. Q.E.D.

We now notice that the above lemma is valid with auxiliary parameters since the proof goes through without change. Thus the subset $S_{\mathbb{R}f} \subset \tilde{M}_{\mathbb{R}f}$ of real lines tangent to the contact distributions of the complex manifolds $\hat{Z}_{tf}$ is a smooth, closed hypersurface transverse to the fibers of $M_{\mathbb{R}f} \xrightarrow{t} \mathbb{R}$. Moreover, S_{0f} is just the sphere $S^{4n-1} = \partial(\mathbb{H}\mathscr{H}_n) \subset \mathbb{HP}_n$. By replacing $\tilde{M}_{\mathbb{R}f} \cong \mathbb{R}^{4n} \times I$ with a smaller neighborhood of $\mathbb{H}\mathscr{H}_n$ (say a ball in $\mathbb{R}^{4n}$ times some smaller interval), we can therefore arrange that each fiber of the projection $\tilde{M}_{\mathbb{R}f} \to I$ meets $S_{\mathbb{R}f}$ in a diffeomorphic copy S_{tf} of S^{4n-1}. Indeed, we only need to choose our neighborhood so that the map $S_{\mathbb{R}f} \to I$ is *proper*.

Now the generalized Jordan curve theorem guarantees that $\tilde{M}_{tf}$ is separated into two components by S_{tf}; moreover, the bounded component M_{tf} is easily seen to be diffeomorphic to $\mathbb{R}^{4n}$ by lifting the vector field d/dt on $\mathbb{R}$ to $\tilde{M}_{\mathbb{R}f} \to I$ in such a way that its flow preserves $S_{\mathbb{R}f}$.

To conclude we have associated to each "sufficiently small' f a manifold M_f diffeomorphic to $\mathbb{R}^n$, defined as a certain set of real twistor lines in $\tilde{Z}_f$. In the next section we will show that M_f carries a natural complete quaternionic-Kähler metric.

5. The deformed metric. We will now produce a complete quaterionic-Kähler metric on the manifold M_f defined in the last section. This metric is the output of the following machine.

THEOREM 1. *Let Z be a complex contact manifold with an antiholomorphic involution which preserves the contact structure. Let $\tilde{M}$ be the space of real twistor lines in Z and let M be a connected component of the subset of real twistor lines which are transverse to the contact distribution. Assume that $M \subset \tilde{M}$ is precompact and that the pseudo-Riemannian quaternionic-Kähler metric g on M defined by the inverse Salamon construction* [16] (*see also* [18], [3]) *has Riemannian signature. Then* (M, g) *is a complete quaternionic-Kähler manifold.*

Proof. Let S be the boundary of M in $\tilde{M}$. (By Proposition 1, $S \subset \tilde{M}$ is a smooth hypersurface.) We need to show that the limit point in $\tilde{M}$ of a Cauchy sequence in (M, g) is never an element of S. For this it suffices to show that every sequence $\{x_j\}$ in M that converges to a point x_∞ of S has divergent distance from any given point $x_0 \in M$.

Let us now recall the construction of the pseudo-Riemannian quaternionic-Kähler metric given in [16]. Let $\tilde{\mathscr{M}}$ be the space of all (complex) twistor lines in Z and let $\mathscr{M}$ denote the open subset consisting of these lines which are transverse to the contact distribution $D = \Theta^\perp \subset TZ$. We may define two vector bundles $E \to \tilde{\mathscr{M}}$ and $H \to \tilde{\mathscr{M}}$ as follows. Let $x \in \tilde{\mathscr{M}}$ correspond to the twistor line $C_x \subset Z$ with normal bundle N_x and let $L \to Z$ denote the contact line bundle $K^{-1/(n+1)}$. Suppose, if necessary by restricting to a neighborhood of a twistor line, that L admits a square-root $L^{1/2} = K^{-1/2(n+1)}$. Then

$$H_x = \Gamma(C_x, L^{1/2})$$

and

$$E_x = \Gamma(C_x, L^{-1/2} \otimes N_x).$$

Since $L^{-1/2} \otimes N_x$ is a trivial bundle on $C_x \cong \mathbb{P}_1$, we have

$$E_x \otimes H_x = \Gamma(C_x, N_x) = T_x\mathscr{M}.$$

We now define a complex-Riemannian metric $\mathfrak{g}$ on $\mathscr{M}$ by defining symplectic forms ω_E and ω_H on the above bundles $E, H \to \mathscr{M} \subset \tilde{\mathscr{M}}$ and then setting $\mathfrak{g} = \omega_E \otimes \omega_H \in \Gamma(\Lambda^2 E^* \otimes \Lambda^2 H^*) \subset \Gamma(\odot^2 T^*\mathscr{M})$. Namely, for $x \in \mathscr{M}$, the normal bundle N_x is canonically identified with the restriction $D|_{C_x}$ of the contact distribution to C_x, and $d\Theta \in \Gamma(L \otimes \Lambda^2 D^*) = \Gamma(\Lambda^2(L^{1/2} \otimes D^*))$. We may therefore define

$$\omega_{E|x} = d\Theta \in \Lambda^2\Gamma(C_x, L^{1/2} \otimes N^*) = \Lambda^2 E_x^*.$$

On the other hand $H_x = \Gamma(C_x, L^{1/2})$ carries the *Wronskian*

$$W: \Lambda^2\Gamma(C_x, L^{1/2}) \to \Gamma(C_x, \Omega^1_{C_x} \otimes L)$$

$$u \wedge v \mapsto u \otimes dv - v \otimes du.$$

But the restriction of the contact form Θ to C_x, where $x \in \mathscr{M}$, automatically yields an isomorphism

$$\Theta_x^{-1}: \Gamma(C_x, \Omega^1_{C_x} \otimes L) \to \mathbb{C}$$

which sends $\Theta|_{C_x}$ to 1. We may then set

$$\omega_H := \Theta_x^{-1} \circ W.$$

The resulting complex-Riemannian metric $\mathfrak{g} := \omega_E \otimes \omega_H$ then [16] has holonomy $\subset \mathrm{Sp}(n, \mathbb{C}) \otimes \mathrm{Sp}(1, \mathbb{C})/\mathbb{Z}_2$. Its restriction g to the real slice $M \subset \mathscr{M}$ therefore is a pseudo-Riemannian metric with holonomy $\subset \mathrm{Sp}(n-l, l) \times \mathrm{Sp}(1)/\mathbb{Z}_2$ for some integer $0 \leqslant l \leqslant n$.

Let us now restate the above construction in a way more suited to boundary considerations. First of all, we have a well-defined line-bundle-valued holomorphic 3-form

$$\Theta \wedge d\Theta \in \Gamma(\tilde{Z}, \Omega^3 \otimes L^2),$$

and we may exploit this by restriction to C_x, $x \in \tilde{\mathscr{M}}$, to define

$$\hat{\omega}_E \in \Gamma(\tilde{\mathscr{M}}, \mathscr{L} \otimes \Lambda^2 E^*)$$

where, as in Proposition 1,

$$\mathscr{L}_x := \Gamma(C_x, \Omega^1_{C_x} \otimes L);$$

namely, we define

$$\begin{aligned}\hat{\omega}_E|_x &= \Theta \wedge d\Theta \in \Gamma(C_x, \Omega^1_{C_x} \otimes \Lambda^2 N^*_x \otimes L^2)\\ &= \Gamma(C_x, \Omega^1_{C_x} \otimes L) \otimes \Lambda^2 \Gamma(C_x, N^*_x \otimes L^{1/2}).\end{aligned}$$

Second, we define $\hat{\omega}_H \in \Gamma(\tilde{\mathscr{M}}, \mathscr{L} \otimes \Lambda^2 H^*)$ to be the Wronskian W

$$\hat{\omega}_{H|x} := W\colon \Lambda^2 \Gamma(C_x, L^{1/2}) \to \Gamma(C_x, \Omega^1_{C_x} \otimes L).$$

Finally, let us recall that, as in Proposition 1, the restriction of Θ to each twistor-like C_x yields a section $\hat{\Theta} \in \Gamma(\tilde{\mathscr{M}}, \mathscr{L})$ with a simple zero along the set $\mathscr{S} \subset \tilde{\mathscr{M}}$ of twistor-likes tangent to D. If we now define

$$\hat{\mathfrak{g}} := \hat{\omega}_E \otimes \hat{\omega}_H \in \Gamma(\tilde{\mathscr{M}}, \mathscr{L}^2 \otimes \Lambda^2 E^* \otimes \Lambda^2 H^*) \subset \Gamma(\tilde{\mathscr{M}}, \mathscr{L}^2 \otimes \Lambda^2 T^* \tilde{\mathscr{M}}),$$

we then observe that $\hat{\mathfrak{g}}$ is related to the previously described complex Riemannian metric on $\mathscr{M} \subset \tilde{\mathscr{M}}$ by

$$\hat{\mathfrak{g}} = \hat{\Theta}^2 \otimes \mathfrak{g}.$$

It therefore follows that $\mathfrak{g}$ times any function which vanishes along $\mathscr{S}$ to order two extends holomorphically across $\mathscr{S}$ and that the conformal class of $\mathfrak{g}$ thus extends across $\mathscr{S}$ in a suitable sense. However, the rank of $\hat{\mathfrak{g}}$ actually drops at $\mathscr{S}$; while $\hat{\omega}_H$ is everywhere nondegenerate, $\hat{\omega}_E$ only has rank 2 at $\mathscr{S}$ (its two nonzero directions being given by $\Theta \in N^* \otimes L$ and $TC_{x^-} \lrcorner d\Theta \in \Omega^1_{C_x} \otimes (D/TC_x)^* \otimes L$), so that $\hat{\mathfrak{g}}$ has rank 4 along $\mathscr{S}$, as opposed to rank $4n$ everywhere else.

On the real slice $\tilde{M} \subset \tilde{\mathscr{M}}$, we may trivialize the line bundle $\mathscr{L}$. Thus, for any defining function α of $S \subset \tilde{\mathscr{M}}$, the quaternionic-Kähler metric g has the property that $\alpha^2 g$ extends real-analytically across S as a tensor field $\hat{g}$ whose rank at S is 4.

Let us consider for a moment the image of $\hat{g}$ in the cotangent space $T^*\tilde{M}$ along S. Our isomorphism $T\tilde{\mathscr{M}} = E \otimes H$ reduces the structure group of $\tilde{M}$ to $\mathrm{GL}(n, \mathbb{H}) \times \mathrm{Sp}(1)/\mathbb{Z}_2$. Since the nondegenerate directions of $\hat{\mathfrak{g}}$ are given by a subspace of E tensor all of H, we conclude that this rank-4 subspace is the "quaternionic span" of some direction with respect to the Sp(1) factor. In fact it must be the quaternionic span of the conormal bundle of $S \subset \tilde{M}$ because $\hat{\omega}_E$ is nonzero in the direction of Θ along $\mathscr{S}$, and our proof of Proposition 1 that $\hat{\Theta}$ has nonzero derivative along $\mathscr{S}$ showed the following exactly: that an element of $\Gamma(C_x, N_x) = \Gamma(C_x, N_x \otimes L^{-1/2}) \otimes \Gamma(C_x, L^{1/2})$ of the form $\nu \otimes \mu$ is transverse to $\mathscr{S}$, whenever $\nu \in \Gamma(C_x, N_x \otimes L^{-1/2}) = E_x$ and $\mu \in \Gamma(C_x, L^{1/2}) = H_x$ are such that $\nu \lrcorner \Theta \in \Gamma(C_x, L^{1/2})$ and μ have their zeros at different places. In other words contraction with Θ gives a map $E_x \to H_x$ when $x \in \mathscr{S}$, and projection from $T_x\tilde{\mathscr{M}}$ to the normal bundle of $\mathscr{S} \subset \tilde{\mathscr{M}}$ is given by the composition

$$E_x \otimes H_x \xrightarrow{\Theta \otimes id} H_x \otimes H_x \xrightarrow{\wedge} \Lambda^2 H_x .$$

It follows that $\hat{g}$ strictly dominates some constant multiple of $d\alpha^2$ near any point of $S \subset \overline{M}$. By compactness there is some constant k such that $\hat{g}$ dominates $kd\alpha^2$ on all of $\overline{M} \subset \tilde{\mathscr{M}}$, since $\hat{g}$ is positive-definite on M and therefore also dominates some multiple of $d\alpha^2$ near any point of the interior $M \subset \overline{M}$. Thus $g > k\, d\alpha^2/\alpha^2$ on all of M. The distance between two points x, $x' \in M$ therefore always exceeds $k|\log \alpha(x) - \log \alpha(x')|$. Since $\alpha = 0$ at S, it follows that the g-distance between a given point $x_0 \in M$ and a sequence of points x_j in M converging to $x_\infty \in S$ must diverge, so that such a sequence x_j is never Cauchy with respect to g. Q.E.D.

COROLLARY. *For f small, the twistor construction gives a complete quaternionic-Kähler metric on each of the manifolds M_f produced in the last section.*

Proof. It suffices to check that the metric is positive-definite. But this pseudo-Riemannian metric is obtained by deforming the symmetric-space metric on $\mathbb{H}\mathscr{H}_n$ through pseudo-Riemannian metrics. But such a deformation leaves the signature of the metric unchanged. Q.E.D.

It remains to show that our construction actually produces metrics on the various manifolds M_f which are geometrically distinct. This will be our task in the next section.

6. The Kodaira-Spencer obstruction. In the previous section, we demonstrated that each of the complex manifolds $\tilde{Z}_f$, where f is any sufficiently small holomorphic section of the contact line bundle L on a neighborhood $\mathscr{U}$ of $U_1 \cap U_3 \subset \mathbb{CP}_{2n+1}$, gives rise to a complete quaternionic-Kähler manifold M_f. In this section we will show that this implies that the moduli space of complete quaternionic-Kähler metrics on $\mathbb{R}^{4n}$ is infinite dimensional. (In order to keep the discussion as simple as possible, let us agree that the latter just means "not finite dimensional".)

Associated with any 1-parameter family Z_t of complex manifolds is the *Kodaira-Spencer obstruction* to the triviality of the family. (For a family of *compact* manifolds to be locally trivial, it is necessary and sufficient for the obstruction to vanish [13]; in the noncompact case, it is merely necessary.) This obstruction is an element of $H^1(Z_t, TZ_t)$ for each t and may concretely be calculated in terms of Cech cohomology as follows. If Z_t can be constructed by first covering Z_0 with open sets U_α and then replacing the identity map $U_\alpha \cap U_\beta \to U_\beta \cap U_\alpha$ with a transition function $\Phi_{\alpha\beta}(t)$, where $\Phi_{\alpha\beta}$ depends smoothly on t and satisfies $\Phi_{\alpha\beta}(0) = id$, $\Phi_{\alpha\beta}(t) = [\Phi_{\beta\alpha}(t)]^{-1}$ and $\Phi_{\alpha\beta}(t)\Phi_{\beta\gamma}(t)\Phi_{\gamma\alpha}(t) = id$, then the assignment of the vector field

$$V_{\alpha\beta} = \frac{d}{dt}\Phi_{\alpha\beta}(t)|_{t=0}$$

to the set $U_\alpha \cap U_\beta$ gives a Cech 1-cocycle with values in $\mathcal{O}(TZ_0)$. The class

$$[V_{\alpha\beta}] = H^1(Z_0, \mathcal{O}(TZ_0))$$

is the Kodaira-Spencer obstruction for $t = 0$. Similarly, $[V_{\alpha\beta}(t)] = [\Phi_{\beta\alpha}\, d/dt\, \Phi_{\alpha\beta}] \in H^1(Z_t, \mathcal{O}(TZ_t))$ is the Kodaira-Spencer obstruction for other values of t.

When Z_t is a complex contact manifold, there is a refined version of this invariant. Indeed, if our identification of regions U_α of Z_0 with regions of Z_t is done in such a manner as to preserve the contact structure, then the vector fields $V_{\alpha\beta}$ will satisfy

$$\mathscr{L}_{V_{\alpha\beta}}\Theta \propto \Theta$$

and are therefore completely characterized by the sections $f_{\alpha\beta} := \Theta(V_{\alpha\beta})$ of the contact line bundle $L = K^{-1/(n+1)}$. Thus we get a contact version of the Kodaira-Spencer obstruction defined by $[f_{\alpha\beta}] \in H^1(Z_t, \mathcal{O}(L_t))$. Since the exact sequence

$$0 \to \mathcal{O}(D_t) \to \mathcal{O}(TZ_t) \to \mathcal{O}(L_t) \to 0$$

canonically splits as a sequence of groups (although not as a sequence of $\mathcal{O}$-modules!), our descent from $H^1(\mathcal{O}(TZ))$ to $H^1(\mathcal{O}(L))$ does not lose any information.

We would now like to measure the nontriviality of our deformations $\tilde{Z}_{tf}$ by using this contact Kodaira-Spencer obstruction. On the other hand, the cohomology group $H^1(\tilde{Z}, \mathcal{O}(L))$ looks rather complicated (at least at first sight). We will get around this problem by restricting the Kodaira-Spencer class to the fourth infinitesimal neighborhoods [10] of twistor lines.

Let us set up the necessary infinitesimal-neighborhood machinery. If $C_x \subset \tilde{Z} \subset \mathbb{CP}_{2n+1}$ is a projective line, let $\mathscr{I} \subset \mathcal{O}_{\mathbb{CP}_{2n+1}}$ denote the ideal of holomorphic functions vanishing on C_x. For each nonnegative integer m, let $\mathcal{O}_{(m)}(L) := \mathcal{O}(L)/[\mathscr{I}^{m+1} \cdot \mathcal{O}(L)]$. There is then a natural restriction map

$$H^1(\tilde{Z}, \mathcal{O}(L)) \to H^1(C_x, \mathcal{O}_{(m)}(L))$$

for any value of m. Moreover, there are exact sequences

$$0 \to \mathcal{O}(L \otimes \odot^m N_x^*) \to \mathcal{O}_{(m)}(L) \to \mathcal{O}_{(m-1)}(L) \to 0$$

of sheaves on C_x. Writing the normal bundle N_x as $E_x \otimes \mathcal{O}(1)$, where $E_x \cong \mathbb{C}^{2n}$ is again defined by $\Gamma(C, N_x \otimes \mathcal{O}(-1))$, this sequence becomes

$$O \to \mathcal{O}(2 - m) \otimes \odot^m E_x^* \to \mathcal{O}_{(m)}(L) \to \mathcal{O}_{(m-1)}(L) \to 0.$$

Thus $H^1(C_x, \mathcal{O}_{(m)}(L)) = 0$ if $m \leqslant 3$, whereas $H^1(C_x, \mathcal{O}_{(4)}(L)) = \odot^4 E_x^*$. The latter isomorphism is, moreover, canonical, provided that C_x is transverse to the contact structure, so that $\Theta|_{C_x}$ provides a basis for

$$H^0(C_x, \Omega^1(2)) = [H^1(C_x, \mathcal{O}(-2))]^*.$$

By construction, the contact Kodaira-Spencer obstruction of the family $\tilde{Z}_{tf}$ at $\tilde{Z}_0 = \tilde{Z} \subset \mathbb{CP}_{2n+1}$ is given by the cocycle

$$f_{13} = f \in \Gamma(U_1 \cap U_3, \mathcal{O}(L)),$$

$$f_{23} = \sigma^* f \in \Gamma(U_2 \cap U_3, \mathcal{O}(L)).$$

(This is a Cech cocycle for the cover $\{U_1, U_2, U_3\}$ precisely because $U_1 \cap U_3$ and $U_2 \cap U_3$ are the only nonempty overlaps; i.e., the cocycle condition $f_{12} + f_{23} + f_{31} = 0$ on $U_1 \cap U_2 \cap U_3$ is vacuous.) Let us calculate the image of this cocycle in $H^1(C_x, \mathcal{O}_{(4)}(L))$.

To do this, notice that the canonical isomorphism $H^1(C_x, \mathcal{O}_{(4)}(L)) = H^1(C_x, \mathcal{O}(L) \otimes \odot^4 N_x^*) \cong \odot^4 E_x^*$ is given by taking the fourth normal derivatives of a Cech representative. The answer can therefore be calculated by a Penrose-type contour integral [9] [20].

Namely, $C_x \cap U_j$ is a Stein cover for $C_x \cong \mathbb{CP}_1$. Choose a closed curve γ_1 in $C_x \cap U_1 \cap U_3$ and a closed curve γ_2 in $C_x \cap U_2 \cap U_3$ as in Figure 1.

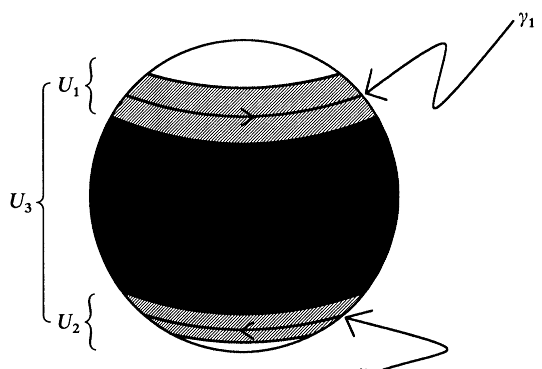

FIGURE 1

Then the isomorphism

$$H^1(C_x, \mathcal{O}(-2)) \xrightarrow{\cong} \mathbb{C}$$

may be explicitly realized by

$$[(g_{13}, g_{23})] \mapsto \frac{1}{2\pi i} \oint_{\gamma_1} g_{13}\Theta + \frac{1}{2\pi i} \oint_{\gamma_2} g_{23}\Theta$$

where $g_{jk} \in \Gamma(C_x \cap U_j \cap U_k, \mathcal{O}(-2))$, since this expression vanishes when $g_{13} = h_1 - h_3$, $g_{23} = h_2 - h_3$ for some $h_j \in \Gamma(C_x \cap U_j, \mathcal{O}(-2))$, but does not vanish for all (g_{13}, g_{23}). (Again, we assume that C_x is transverse to the contact structure.)

Thus, if f is any function on a neighborhood of $U_1 \cap U_3$, the contact Kodaira-Spencer obstruction of the family is detected by the contour integral

$$\Psi_{ABCD}(x) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{\partial^4 f}{\partial z^A \partial z^B \partial z^C \partial z^D} \Theta + \frac{1}{2\pi i} \oint_{\gamma_2} \frac{\partial^4 \overline{\sigma^* f}}{\partial z^A \cdots \partial z^D} \Theta$$

where $A, \ldots, D$ range over $1, \ldots, 2n$. Here, f is considered as a function of homogeneity 2 on $\mathbb{C}^{2n+2}$, and

$$(\sigma^* f)(z_1, z_2, \ldots, z_{2n+1}, z_{2n+2}) := f(-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2n+2}, \bar{z}_{2n+1}).$$

If we assume that C_x is a *real* twistor line, we may take $\gamma_2 = \sigma\gamma_1$ and obtain

$$\Psi_{ABCD}(x) = \frac{1}{2\pi i} \oint_{\gamma_1} \left[\frac{\partial^4 f}{\partial z^A \cdots \partial z^D} \Theta + \frac{\partial^4 \bar{f}}{\partial \hat{z}^A \cdots \partial \hat{z}^D} \bar{\Theta} \right]$$

where

$$(\hat{z}^1, \hat{z}^2, \ldots, \hat{z}^{2n+1}, \hat{z}^{2n+2}) = (-\bar{z}^2, \bar{z}^1, \ldots, -\bar{z}^{2n+2}, \bar{z}^{2n+1}).$$

Thus, if we choose

$$f(z_1, \ldots, z_{2n+1}) = F\left[\frac{z_1}{z_{2n+2}}, \ldots, \frac{z_{2n}}{z_{2n+2}}\right] \frac{z_{2n+2}^3}{z_{2n+1}}$$

for $F(\zeta_1, \ldots, \zeta_{2n})$ a holomorphic function in a ball of radius $>(1 + \varepsilon)$ and take γ_1 to be $|z_1|^2 = (\varepsilon + \delta)|z_2|^2/2$, we obtain

$$\Psi_{ABCD}(x) = \left(1 - \sum_{j=1}^{2n} |z^j|^2\right) \left[\frac{\partial^4 F}{\partial z^A \cdots \partial z^D} + \frac{\partial^4 \bar{F}}{\partial \hat{z}^A \cdots \partial \hat{z}^D} \right]$$

for $x = (q^1, \ldots, q^n) = (z^1, \ldots, z^{2n})$ in the unit ball. In particular the space of functions F depending only on the even-numbered variables $(z^2, z^4, \ldots, z^{2n})$, defined on a ball of radius $>(1 + \varepsilon)$ in $\mathbb{C}^n$, and vanishing to order 3 at the origin, injects into $H^1(\tilde{Z}, \mathcal{O}(L))$.

We will now see that the geometry of the associated quaternionic-Kähler manifolds M_{tf} is correspondingly altered. To do this let us ask whether the third infinitesimal neighborhood of a twistor line in Z can be isomorphic to the third infinitesimal neighborhood of a "nearby" twistor line in $\tilde{Z}_{tf}$. Such an infinitesimal neighborhood defines an element of $H^1(\mathbb{CP}_1, \mathrm{Aut}(\mathcal{O}_{(3)}))$, where $\mathcal{O}_{(3)}$ is the structure sheaf of the third infinitesimal neighborhood of $\mathbb{CP}_1 \subset \mathbb{CP}_{2n+1}$. One can show [8] that this torsor is given by

$$H^1(\mathbb{CP}_1, \mathrm{Aut}(\mathcal{O}_{(3)})) = H^1(\mathbb{CP}_1, (T\mathbb{CP}_{2n+1}|_{\mathbb{CP}_1}) \otimes \odot^3 N^*),$$

as follows from the fact that there are central extensions

$$\mathrm{Der}(\mathcal{O}_{(m)}, \mathcal{I}^m/\mathcal{I}^{m+1}) \rightarrowtail \mathrm{Aut}(\mathcal{O}_{(m)}) \twoheadrightarrow \mathrm{Aut}(\mathcal{O}_{(m-1)})$$

essentially by a nonlinear version of the long-exact sequence associated with a short-exact sequence. (Here the automorphism of $\mathcal{O}_{(m)}$ associated with a derivation $v \in \mathrm{Der}(\mathcal{O}_{(m)}, \mathcal{I}^m/\mathcal{I}^{m+1}) \cong \mathcal{O}(T\mathbb{CP}_{2n+1} \otimes \odot^m N^*)$, $m \geqslant 1$, is just $1 + v$.) On the other hand, if our line $\mathbb{CP}_1 \subset \mathbb{CP}_{2n+1}$ is transverse to the standard contact distribution, we have an injective linear map

$$(d\Theta)^{-1}\colon \odot^4 N^* \otimes \mathcal{O}(2) \hookrightarrow \odot^3 N^* \otimes D$$

obtained by identifying the normal bundle N with D and remembering that $d\Theta$: $D \to D^* \otimes \mathcal{O}(2)$ is an isomorphism. This results in an injection $H^1(\mathbb{CP}_1, \mathcal{O}_{(4)}(2)) \hookrightarrow H^1(\mathbb{CP}_1, \mathrm{Aut}(\mathcal{O}_{(3)}))$, $\odot^4 E^* \hookrightarrow E \otimes \odot^3 E^*$, which is none other than the restriction of the splitting $H^1(Z, \mathcal{O}(2)) \to H^1(Z, \mathcal{O}(TZ))$ to an infinitesimal neighborhood of $\mathbb{CP}_1$. The upshot is that our deformations $\tilde{Z}_{tf}$ *effectively deform the third infinitesimal neighborhoods of twistor lines* as F ranges over the given space of holomorphic functions. But since the local geometry of M_{tf} near x determines the biholomorphism type of the germ of $C_x \subset \tilde{Z}_{tf}$, it follows that the deformations M_{tf} are all distinct as F ranges over the holomorphic functions in a ball in $\mathbb{C}^n$ which vanish to order 3 at 0; i.e., we have given an infinite-dimensional family of effective deformation of the quaternionic-Kähler manifold $\mathbb{H}\mathcal{H}_n$ through quaternionic-Kähler manifolds. Since the isometry group of $\mathbb{H}\mathcal{H}_n$ is, of course, finite dimensional, we have proved the following theorem.

MAIN THEOREM. *The moduli space of complete quaternionic-Kähler metrics on $\mathbb{R}^{4n}$ is infinite dimensional.*

Remark. Our proof shows that Ψ_{ABCD} measures a (paraconformally invariant) change in the geometry. In fact, ignoring the factor of $(1 - \Sigma|z^j|^2)$, which may be viewed as corresponding to a paraconformal weight, Ψ actually corresponds to the t-derivative of the piece of the curvature tensor which lives in

$$(\odot^4 E^*) \otimes (\Lambda^2 H^*)^2 \subset (\Lambda^2 T^* M) \odot (\Lambda^2 T^* M).$$

7. Concluding remarks. While we have focused on the higher-dimensional case and the associated holonomy problem, our theorem shows that the space of complete self-dual Einstein metrics on $\mathbb{R}^4$ is infinite dimensional. In fact the proof gives more—namely, there is an infinite-dimensional space of conformal metrics on S^3

which bound complete self-dual Einstein metrics on the 4-ball, in the sense that the conformal structure is smooth up to the boundary, making it into a *conformal infinity*. Let us point out the relationship between this and earlier results.

First off, the present author [14] proved some time ago that any real-analytic conformal metric on a 3-manifold is *locally* the conformal infinity of a unique self-dual Einstein 4-manifold. Pedersen [7] then produced an explicit 1-parameter family of (left-invariant) conformal metrics on S^3 which bound complete metrics on the 4-ball.

By contrast, Graham and Lee [11] have recently proved (by the inverse function theorem) that any conformal metric on S^{m-1} sufficiently close to the standard one is the conformal infinity of a complete Einstein metric on the ball. It is therefore natural to ask which conformal metrics on S^3 bound a complete *self-dual* Einstein metric.

An analogous problem arises when one examines the Dirichlet problem on the 2-dimensional disk. Any smooth complex-valued function on the circle is the boundary value of a harmonic function on the disk, but only a subclass of functions, those of "positive frequency", are the boundary values of holomorphic functions, whereas a complementary set of ("negative frequency") functions are boundary values of antiholomorphic functions.

Let us define a conformal metric $[h]$ on S^3 to be of *positive frequency* if it is the conformal infinity of a complete *self-dual* Einstein metric on the 4-ball; similarly, define it to be of *negative frequency* if it is the conformal infinity of an anti-self-dual Einstein metric. The following then seems most natural.

Positive Frequency Conjecture. Any conformal metric $[h]$ on S^3 which is sufficiently near the standard conformal metric h_0 can be expressed in the form

$$h = \alpha^2 \varphi^*(h_0 + h_+ + h_-)$$

where $\varphi\colon S^3 \to S^3$ is a diffeomorphism, α is a nonzero function, h_+ and h_- are trace-free symmetric tensor fields, and $h_0 + h_+$ is of positive frequency, while $h_0 + h_-$ is of negative frequency.

Such a result would then seem to provide a natural polarization for quantum gravity, at least on the level of scattering theory.

References

1. M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461.
2. T. N. Bailey and M. G. Eastwood, *Complex paraconformal manifolds—their differential geometry and twistor theory*, Forum Math. 3 (1991), 61–103.
3. L. Berard-Bergery, *Une généralisation de la théorie des twisteurs de Penrose*, unpublished lectures at the Universite de Nancy I, 1983.
4. M. Berger, *Remarques sur le groupe d'holonomie des variétés Riemanniennes*, C. R. Acad. Sci. Paris **262** (1966), 1316–1318.
5. A. L. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.
6. R. L. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. **126** (1987), 525–576.

7. A. Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier **16** (1966), 1–95.
8. M. G. Eastwood and C. R. LeBrun, *Fattening complex manifolds, curvature, and Kodaira-Spencer maps*, preprint.
9. M. G. Eastwood, R. Penrose and R. O. Wells Jr., *Cohomology and massless fields*, Comm. Math. Phys. **78** (1981), 305–351.
10. P. Griffiths, *The extension problem in complex analysis II: embeddings with positive normal bundle*, Amer. J. Math. **88** (1966), 366–446.
11. C. R. Graham and J. Lee, *Einstein metrics on the ball with prescribed conformal infinity*, preprint.
12. K. Kodaira, *On the stability of compact submanifolds of complex manifolds*, Amer. J. Math. **85** (1963), 79–94.
13. K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures I & II*, Ann. of Math. **67** (1958), 328–466.
14. C. R. LeBrun, *$\mathscr{H}$-space with a cosmological constant*, Proc. Roy. Soc. London Ser. A **380** (1982), 171–185.
15. ———, *A rigidity theorem for quaternionic-Kähler manifolds*, Proc. Amer. Math. Soc. **103** (1988), 1205–1208.
16. ———, *Quaternionic-Kähler manifolds and conformal geometry*, Math. Ann. **284** (1989), 353–376.
17. H. Pedersen, *Einstein metrics, spinning top motions and monopoles*, Math. Ann. **274** (1986), 35–59.
18. H. Pedersen and Y.-S. Poon, "Twistorial construction of quaternionic-Kähler manifolds" in *Proc. of the VIth International Colloquium on Differential Geometry*, Cursos y congresos **61**, Universidad de Santiago de Compostela, Spain, 1989, 207–218.
19. R. Penrose, *Non-linear gravitons and curved twistor theory*, Gen. Relativity Gravitation **7** (1976), 31–52.
20. R. Penrose and W. Rindler, *Spinors and Space-time, Volume* 2, Cambridge Univ. Press, Cambridge, 1986.
21. Y.-S. Poon and S. Salamon, *Quaternionic-Kähler eight-manifolds with positive scalar curvature*, J. Differential Geometry **33** (1991), 363–378.
22. M. Namba, *On maximal families of compact complex manifolds*, Tohoku Math. J. **24** (1972), 581–609.
23. S. Salamon, *Riemannian Geometry and Holonomy Groups*, Longman, Harlow, Essex, 1989.
24. ———, *Quaternionic-Kähler manifolds*, Invent. Math. **67** (1982), 143–171.
25. M. Thornber, *Vanishing theorems for quaternionic-Kähler manifolds*, Ph.D. thesis, State Univ. of New York at Stony Brook, 1989.
26. R. S. Ward, *Self-dual space-times with cosmological constant*, Comm. Math. Phys. **78** (1980), 1–17.

Max-Planck-Institut für Mathematik, Gottfried-Claren-Straße 26, 5300 Bonn 3, Germany

Current: Department of Mathematics, State University of New York, Stony Brook, New York 11794-3651