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# Scalar-flat Kähler metrics on blown-up ruled surfaces

By *Claude LeBrun* [1]) at Stony Brook

## § 1. Introduction

Let $(M, J)$ be a compact complex surface (i.e. a complex 2-manifold). A Kähler metric $g$ on $M$ will be said to be *scalar-flat* if its scalar curvature $R$ is identically zero.

There are a number of compelling reasons for studying scalar-flat Kähler surfaces. These include the following:

- Such manifolds have anti-self-dual Weyl curvature [9]; indeed [12], they are the only anti-self-dual compact Riemannian 4-manifolds with non-negative scalar curvature and indefinite intersection form. The Penrose twistor correspondence [16], [1] then relates them to the theory of complex 3-manifolds. Moreover, they are therefore absolute minima of the conformally invariant functional $\int |W^i{}_{jkl}|^2 \, d\mathrm{vol}$, which is of independent interest because its critical points also include all Einstein metrics [2].

- Such metrics are *critical* in the sense of Calabi [6] – in fact, they are absolute minima of the functionals $\int R^2 \, d\mathrm{vol}$, $\int |R_{ij}|^2 \, d\mathrm{vol}$, and $\int |R^i_{jkl}|^2 \, d\mathrm{vol}$ for metrics in a fixed Kähler class.

- As they include all Ricci-flat Kähler surfaces, the classification problem for scalar-flat Kähler surfaces may be considered a natural extension of the Calabi conjecture solved by Yau [19], [20].

- The unusual conformal geometry of such manifolds makes them exceptional for the theory of harmonic maps between Kähler manifolds [5].

Despite these compelling features of the problem, the subject has suffered from a genuine dearth of examples. Using a plurigenera vanishing theorem of Yau [18] and the

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Enriques-Kodaira classification, it is immediate that a scalar-flat Kähler surface must either be Ricci-flat, rational, or ruled, and one may then show [3], [12] that, if the surface is assumed to be non-Ricci-flat and *relatively minimal* – i.e. not obtained from another complex surface by the $\sigma$-process ("blowing up") – it is necessarily a quotient of the conformally flat symmetric space $S^2 \times \mathscr{H}^2$ by an isometric action of the fundamental group $\Gamma$ of a compact Riemann surface. Until now, there have been no other examples except for the Ricci-flat ones. As it had been shown [4] that there are obstructions to the existence of scalar-flat Kähler metrics which are more subtle than the Futaki invariant [8], there seemed to be ample reason for pessimism regarding the construction of other scalar-flat Kähler surfaces.

We will now remedy this by producing explicit examples of scalar-flat Kähler metrics on *blow-ups* of ruled surfaces. These new metrics are neither Ricci-flat nor conformally flat, since $c_1^2 \neq 0$ and $p_1 \neq 0$ for a blown-up ruled surface. Our approach, which is limited to metrics with an isometric $S^1$-action, exploits a variant of the generalized Gibbons-Hawking ansatz recently described by the author in [13]. Despite the limitation of the method to metrics with a Killing field, it nonetheless produces scalar-flat Kähler surfaces in most deformation classes not excluded by Yau's above-cited theorem; one may therefore hope that the current examples will serve as a serviceable beach-head from whence to begin the next phase of the assault on the general case.

The main-spring of the present construction is the ansatz presented in § 2, which produces scalar-flat Kähler surfaces from positive harmonic functions on hyperbolic 3-space. The harmonic functions we will use in practice are built up from the Green's functions of Fuchsian groups, and the next item on the agenda (§ 3) is therefore a self-contained construction of these fundamental solutions in terms of Poincaré series. In §§ 4–5, we then yoke these beasts of burden to our machinery, and proceed to bring in the promised harvest of compact Kähler surfaces.

If we choose to forget about complex structures and reverse the orientation, the present construction yields self-dual conformal metrics on $(S^2 \times S_g) \# \mathbb{C}P_2 \# \ldots \# \mathbb{C}P_2$, where $S_g$ is a Riemann surface of genus $g \geqq 2$. It seems that the existence of such metrics can also be deduced from the connected sum machinery of Donaldson and Friedman [7], although a certain amount of care is needed in order to check that the twistor space $Z$ of a generic conformally-flat metric on $(S^2 \times S_g)$ satisfies $H^2(Z, \mathcal{O}(TZ)) = 0$.

This article is the third in a series (cf. [13], [14]) in which explicit $\pm$ self-dual metrics on compact manifolds are constructed from hyperbolic multi-monopoles. The author has nonetheless endeavored to make the current paper completely self-contained.

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## § 2. Metrics, monopoles, and moment-map magic

In this section, we will show that a positive harmonic function $V$ on a region of hyperbolic 3-space $\mathscr{H}^3$ gives rise to a scalar-flat Kähler surface with isometric $S^1$-action. In essence, this is an inverse of the Kähler quotient and moment-map constructions [13]. The overall strategy of the present article is to apply this construction to harmonic functions given as Poicaré series, namely sums of Green's functions ("monopoles") over orbits of Fuchsian groups.

Let $\mathscr{U} = \{(x, y, z) \mid y > 0, -1 < z < 1\}$ denote the region $\mathbb{R} \times \mathbb{R}^+ \times ]-1, 1[$ of $\mathbb{R}^3$, and consider the Riemannian metric

$$h = \frac{dx^2 + dy^2}{(1 - z^2)y^2} + \frac{dz^2}{(1 - z^2)^2}$$

on $\mathscr{U}$. Then $(\mathscr{U}, h)$ is just hyperbolic 3-space $\mathscr{H}^3$ equipped with an unusual coordinate system; if we set $(r, s, t) = (x, yz, y\sqrt{1 - z^2})$, $h$ becomes

$$h = \frac{dr^2 + ds^2 + dt^2}{t^2},$$

yielding the usual upper-half-space model. Notice that the half-plane $z = 0$, $y > 0$, is totally geodesic.

Let $V > 0$ be a function on $\mathscr{W} \subset \mathscr{U}$ which satisfies $\Delta V = 0$, where $\Delta$ is the Laplace-Beltrami operator of the hyperbolic metric $h$ on $\mathscr{U}$; thus we assume that

$$y^2 V_{xx} + y^2 V_{yy} + (1 - z^2) V_{zz} = 0 .$$

Since this is equivalent to the statement that the 2-form $* dV$ is closed, where $*$ is the Hodge star operator of $h$, we may consider the deRham class $\left[\frac{1}{2\pi} * dV\right]$, and it makes sense to impose the extra condition that this be an *integral* class – i.e. in the image of

$$H^2(\mathscr{W}, \mathbb{Z}) \to H^2(\mathscr{W}, \mathbb{R}).$$

With this hypothesis, there is then a circle bundle $\pi : M \to \mathscr{W}$ which carries a connection form $\omega$ with curvature

$$d\omega = * dV.$$

We now equip $M$ with the Riemannian metric

$$\begin{aligned} g &= (1 - z^2)(Vh + V^{-1}\omega^2) \\ &= \frac{V}{y^2}(dx^2 + dy^2) + \frac{V}{1 - z^2} dz^2 + \frac{1 - z^2}{V} \omega^2 \end{aligned}$$

and the almost complex structure $J$ defined by

$$dx \mapsto dy, \quad dz \mapsto \frac{1-z^2}{V}\omega .$$

I then claim that $J$ is integrable, and that $g$ is Kähler with respect to $J$.

Indeed, the differential ideal generated by $dx + i dy$ and $\frac{V dz}{1-z^2} + i\omega$ is closed, since

$$\begin{aligned} d\left[\frac{V dz}{1-z^2} + i\omega\right] &= dV \wedge \frac{dz}{1-z^2} + i * dV \\ &= (V_x dx + V_y dy) \wedge \frac{dz}{1-z^2} \\ &\quad + i\left(V_x \frac{dy \wedge dz}{1-z^2} + V_y \frac{dz \wedge dx}{1-z^2} + V_z \frac{dx \wedge dy}{y^2}\right) \\ &= (dx + i dy) \wedge \left[\frac{V_x - iV_y}{1-z^2} dz + V_z \frac{dy}{y^2}\right]. \end{aligned}$$

Since $g$ is Hermitian with respect to $J$, we may consider its associated 2-form

$$\Omega = \frac{V}{y^2} dx \wedge dy + dz \wedge \omega ,$$

and observe that

$$\begin{aligned} d\Omega &= \frac{dV}{y^2} \wedge dx \wedge dy - dz \wedge * dV \\ &= \frac{V_z}{y^2} dz \wedge dx \wedge dy - dz \wedge \left(\frac{V_z}{y^2} dx \wedge dy\right) \\ &= 0, \end{aligned}$$

so that $(g, J)$ is Kähler.

We now consider the scalar curvature of $g$. This can be done most efficiently by first noticing that, if $(e_1, e_2, e_3, e_4)$ is the frame dual to the coframe $(dx, dy, dz, \omega)$, then $e_4$ is a Killing vector field preserving $\omega$, and hence

$$e_4 - iJe_4 = e_4 + i\frac{(1-z^2)}{V} e_3$$

is a holomorphic vector field; hence, since $dx + i dy$ is a holomorphic (1, 0)-form, there exist holomorphic charts such that the coordinate volume form is $\frac{V}{1-z^2} dx \wedge dy \wedge dz \wedge \omega$. But the volume form of $g$ is

$$\frac{1}{2}\Omega \wedge \Omega = \frac{V}{y^2} dx \wedge dy \wedge dz \wedge \omega,$$

or $\frac{1-z^2}{y^2}$ times the coordinate volume form. The Ricci form is therefore

$$\begin{aligned} P &= -i\partial\bar{\partial}\log\left(\frac{1-z^2}{y^2}\right) \\ &= -\frac{1}{2} dJd[\log(1-z^2) - 2\log y] \\ &= -\frac{1}{2} d\left[\frac{-2zJdz}{1-z^2} - \frac{2Jdy}{y}\right] \\ &= d\left[\frac{z}{V}\omega - \frac{dx}{y}\right] \\ &= d(zV^{-1}\omega) - \frac{dx \wedge dy}{y^2}. \end{aligned}$$

Hence

$$\begin{aligned} \Omega \wedge P &= \frac{V}{y^2} dx \wedge dy \wedge d(zV^{-1}\omega) + dz \wedge \omega \wedge d(zV^{-1}\omega) \\ &\quad - \frac{1}{y^2} dz \wedge \omega \wedge dx \wedge dy \\ &= \frac{V}{y^2}(zV^{-1})_z dx \wedge dy \wedge dz \wedge \omega + zV^{-1} d\omega \wedge dz \wedge \omega \\ &\quad - \frac{1}{y^2} dx \wedge dy \wedge dz \wedge \omega \\ &= \left[\frac{V}{y^2}(zV^{-1})_z + zV^{-1}\frac{V_z}{y^2} - \frac{1}{y^2}\right] dx \wedge dy \wedge dz \wedge \omega \\ &= \left[\frac{1}{y^2}(zVV^{-1})_z - \frac{1}{y^2}\right] dx \wedge dy \wedge dz \wedge \omega \\ &= 0, \end{aligned}$$

showing that the scalar curvature $\langle \Omega, P \rangle = *(\Omega \wedge P)$ of $g$ vanishes.

## § 3. Green's functions and Fuchsian groups

In the previous section, we described a method which constructs scalar-flat Kähler metrics from positive harmonic functions on hyperbolic 3-space. In this section, we give an elementary construction of an important family of such functions, namely the Green's functions of hyperbolic 3-space modulo the fundamental groups of compact Riemann surfaces. For a more general construction of such objects, cf. [17].

Let $\Gamma \subset PSL(2, \mathbb{R}) = SO_+(1,2)$ be the fundamental group of some compact Riemann surface $S_g$ of genus $g \geqq 2$; we will identify $\Gamma$ with its image in $PSL(2, \mathbb{C}) = SO_+(1,3)$ via the tautological inclusion $SL(2, \mathbb{R}) \hookrightarrow SL(2, \mathbb{C})$. The Fuchsian group $\Gamma$ then acts on hyperbolic 3-space in a proper, discrete manner. In fact, this action is easy to understand in terms of our model $\mathcal{U}$ for $\mathcal{H}^3$; if

$$(x, y) \mapsto (\tilde{x}, \tilde{y})$$

is a fractional linear transformation given by

$$\tilde{x} + i\tilde{y} = \frac{a(x+iy)+b}{c(x+iy)+d}$$

for some given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, there is an induced isometry of $(\mathcal{U}, h)$ given by

$$(x, y, z) \mapsto (\tilde{x}, \tilde{y}, z).$$

For $p = (x_0, y_0, z_0) \in \mathcal{U}$ an arbitrary point, we now let $p\Gamma$ denote the orbit of $p$ under $\Gamma = \pi_1(S_g)$. Note that $p\Gamma$ is a discrete subset of a half-plane $z =$ constant, $y > 0$. For $\alpha \in \Gamma$, we denote the image of $p$ under $\alpha$ by $p\alpha$.

Let $\varrho_\alpha : \mathcal{H}^3 \to \mathbb{R}$ denote the hyperbolic distance from $p\alpha$, and let $G_{p\alpha} = [e^{2\varrho_\alpha} - 1]^{-1}$, which is the Green's function of $p\alpha$ with the normalization

$$\Delta G_{p\alpha} = -2\pi\delta_{p\alpha},$$

where $\Delta$ is the Laplace-Beltrami operator on $\mathcal{H}^3$ and $\delta_{p\alpha}$ is the Dirac delta distribution centered at $p\alpha$. We can now try to define the Green's function $G_{p\Gamma}$ of $p\Gamma$ by

$$G_{p\Gamma} := \sum_{\alpha \in \Gamma} G_{p\alpha}. \tag{1}$$

**Lemma 1.** *The series* (1) *converges to a smooth function on* $\mathcal{H}^3 - p\Gamma$. *Moreover,* $G_{p\Gamma}$ *solves the equation*

$$\Delta G_{p\Gamma} = -2\pi \sum_{\alpha \in \Gamma} \delta_{p\alpha}$$

*in the sense of distributions. Finally,* $G_{p\Gamma}$, *considered as a function on* $\mathcal{U} - p\Gamma$, *is smooth up to the boundary at* $z = \pm 1$, $y > 0$, *and vanishes at these surfaces.*

*Proof.* We begin by considering the function $r_\alpha : \mathscr{H}^2 \to \mathbb{R}$ given by the hyperbolic distance from $(x_0, y_0)\alpha$ in the upper half-plane $\mathscr{H}^2 = \mathbb{R} \times \mathbb{R}^+$ with respect to $\frac{dx^2 + dy^2}{y^2}$, and then extend this to $\mathscr{U} = \mathscr{H}^2 \times ]-1, 1[$ as a function independent of $z$. We then have $\varrho_\alpha \geqq r_\alpha$, since $h = \frac{dx^2 + dy^2}{(1 - z^2)y^2} + \frac{dz^2}{(1 - z^2)^2}$ dominates $\frac{dx^2 + dy^2}{y^2}$ term by term. (In fact, $\varrho_\alpha > r_\alpha$ if $z_0 \neq 0$.) Since a fundamental domain $\mathscr{R}$ in $\mathscr{H}^2$ for $\Gamma$ has area equal to

$$\operatorname{area}(S_g) = 4\pi(g - 1),$$

letting the diameter of $\mathscr{R}$ be $D$, we see that the number $n_R$ of elements of $p\Gamma$ within the annulus $R \leqq r \leqq R + 1$, where $r$ is the distance from an arbitrary point in $\mathscr{H}^2$, satisfies

$$n_R \leqq \frac{A_{R+D+1} - A_{R-D}}{4\pi(g - 1)},$$

where $A_R = 4\pi \sinh^2\left(\frac{R}{2}\right)$ is the area of a hyperbolic disk of radius $R$. In particular,

$$n_R < \frac{1}{2} e^{R+D+1}.$$

For some $q \in \mathscr{U}$, let $\mathscr{S}_a^b \subset p\Gamma$ be the subset consisting of points whose projections to $\mathscr{H}^2 = \mathbb{R} \times \mathbb{R}^+$ are contained in a hyperbolic annulus $a \leqq R \leqq b$ centered at the projection of $q$. On the ball of radius 1 about $q \in \mathscr{H}^3$, we therefore have

$$\begin{aligned} \sum_{\alpha \in \mathscr{S}_j^k} G_{p\alpha} &< \sum_{l=j}^{k} \frac{n_l}{e^{2(l-1)} - 1} \\ &< \sum_{l=j}^{k} \frac{e^{l+D+1}}{e^{2l-2}} \\ &< e^{D+4-j} \end{aligned}$$

for $k > j > 1$ any integers. Letting $\mathscr{S}$ denote the (finite) set of $\alpha \in \Gamma$ for which the projection of $p\alpha$ to $\mathscr{H}^2$ is within distance 2 of the projection of $q$, we conclude that

$$\sum_{\alpha \in \Gamma - \mathscr{S}} G_{p\alpha}$$

converges uniformly on a neighborhood of $q$, and so is continuous in a neighborhood of our arbitrary point $q$.

It follows that the series (1) for $G_{p\Gamma}$ defines a continuous function on $\mathscr{H}^3 - p\Gamma$. It also follows that $G_{p\Gamma}$ is a well-defined element of $\mathscr{D}'$, where $\mathscr{D}$ denotes the smooth, compactly supported functions of $\mathscr{H}^3$ with the topology of uniform convergence on compact sets, and that, as such, $G_{p\Gamma}$ is a solution of

$$\Delta G_{p\Gamma} = -2\pi \sum_{\alpha \in \Gamma} \delta_{p\alpha}.$$

Elliptic regularity then guarantees that $G_{p\Gamma}$ is smooth away from $p\Gamma$.

Finally let us examine $G_{p\Gamma}$ at the surfaces $z = \pm 1$, $y > 0$. To do this, notice that $h$ dominates $\frac{dx^2 + dy^2}{y^2} + \frac{dz^2}{(1-z^2)^2}$ term by term. Hence, letting $c = z_0$ denote the $z$-coordinate of $p$,

$$\varrho_\alpha \geqq \sqrt{r_\alpha^2 + \left(\ln \sqrt{\frac{1+z}{1-z}} - \ln \sqrt{\frac{1+c}{1-c}}\right)^2}$$

$$\geqq \frac{r_\alpha}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \left| \ln \left( \frac{1+z}{1-z} \, \frac{1-c}{1+c} \right) \right| .$$

Thus

$$e^{2\varrho_\alpha} \geqq e^{\sqrt{2} r_\alpha} \max \left[ \left( \frac{1+z}{1-z} \, \frac{1-c}{1+c} \right)^{\frac{1}{2\sqrt{2}}}, \left( \frac{1-z}{1+z} \, \frac{1+c}{1-c} \right)^{\frac{1}{2\sqrt{2}}} \right]$$

and

$$e^{2\varrho_\alpha} - 1 > \frac{1}{2} e^{\sqrt{2} r_\alpha} \max \left[ \left( \frac{1+z}{1-z} \, \frac{1-c}{1+c} \right)^{\frac{1}{2\sqrt{2}}}, \left( \frac{1-z}{1+z} \, \frac{1+c}{1-c} \right)^{\frac{1}{2\sqrt{2}}} \right]$$

provided that $|z| > \frac{1+3|c|}{3+|c|}$. Hence

$$\frac{1}{e^{2\varrho_\alpha} - 1} < 2e^{-\sqrt{2} r_\alpha} \min \left[ \left( \frac{1+z}{1-z} \, \frac{1-c}{1+c} \right)^{\frac{1}{2\sqrt{2}}}, \left( \frac{1-z}{1+z} \, \frac{1+c}{1-c} \right)^{\frac{1}{2\sqrt{2}}} \right].$$

But $\sum_{\alpha \in \Gamma} e^{-\sqrt{2} r_\alpha}$ converges by an argument similar to that used for $\sum_{\alpha \in \Gamma} e^{-2r_\alpha}$, because $n_R \sim e^R$ and the exponent here is less than $-r_\alpha$ by a definite factor. We conclude that $G_{p\Gamma}$ is continuous up to the surfaces $z = \pm 1$ and vanishes there. But then a regularity result of Robin Graham ([10], p. 631, Theorem 11.4) guarantees that, by virtue of the fact that $\Delta G_{p\Gamma} = 0$ for $|z| > |c|$, $G_{p\Gamma}$ is in fact smooth up to $z = \pm 1$. QED

## § 4. Blown-up product surfaces

We now use the procedure of § 2 together with the Green's functions constructed in § 3 to produce some specific scalar-flat Kähler manifolds.

Let $p = (a, b, c) \in \mathscr{U}$ with $c > 0$, and set $q = (a, b, -c)$. Let

$$V := 1 + G_{p\Gamma} + G_{q\Gamma}$$

where the Green's function terms are as in § 3. Notice that

$$V(x, y, z) = V(x, y, -z),$$

so that

$$*dV|_{z=0} \equiv 0.$$

Moreover, $\left[\frac{1}{2\pi} * dV\right] \in H^2(\mathscr{U} - p\Gamma - q\Gamma, \mathbb{R})$ is an integral class, since for any compact surface $S \subset \mathscr{U} - p\Gamma - q\Gamma$, we have

$$\begin{aligned}\int_S * dV &= \int_{\mathscr{T}} d * dV \\ &= -2\pi \int_{\mathscr{T}} \sum_{\alpha\in\Gamma} (\delta_{p\alpha} + \delta_{q\alpha}) \\ &= -2\pi \,\#\, [\mathscr{T} \cap (p\Gamma \cup q\Gamma)],\end{aligned}$$

where $\mathscr{T} \subset \mathscr{U}$ is the region inside $S$.

Now let $\pi_0 : M_0 \to \mathscr{U} - p\Gamma - q\Gamma$ denote the circle bundle with connection form $\omega$ such that $d\omega = * dV$; because $\mathscr{U} - p\Gamma - q\Gamma$ is simply connected, this defines $M_0$ uniquely. The construction from § 2 then tells us that

$$g = (1 - z^2)(Vh + V^{-1}\omega^2)$$

is scalar-flat and Kähler.

While the manifold $(M_0, g)$ is incomplete, there is a standard manner of completing it, namely the method of "nuts and bolts" [11]. The idea is to add a single point to $M_0$ for each point in $p\Gamma \cup q\Gamma$, and to add surfaces $\mathbb{R}^2$ along $z = +1$ and $z = -1$. The circle action on $M_0$ will extend to the larger manifold $M \supset M_0$ in such a way that the added isolated points ("nuts") and added surfaces ("bolts") become the fixed points of the action.

In order to carry this out, let us examine the metric $g$ near a point of $p\Gamma \cup q\Gamma$. Introducing exponential polar coordinates on $\mathscr{U}$ centered at this point, our formula for the metric becomes

$$\frac{g}{1 - z^2} = \left(\frac{1}{2r} + F\right)(dr^2 + \sinh^2 r g_{S^2}) + \left(\frac{1}{2r} + F\right)^{-1} \omega^2$$

where $F$ is a smooth function on a neighborhood of $0 \in \mathbb{R}^3$. (Here $g_{S^2}$ denotes the standard metric of the unit 2-sphere.) For $r$ small, the circle bundle $\pi_0 : M_0 \to \mathscr{U} - p\Gamma - q\Gamma$ may be identified with

$$\begin{aligned}\wp : \mathbb{C}^2 &\to \mathbb{R}^3, \\ (Z_1, Z_2) &\mapsto \left(\frac{|Z_1|^2 - |Z_2|^2}{2}, \operatorname{Im} Z_1 \bar{Z}_2, \operatorname{Re} Z_1 \bar{Z}_2\right),\end{aligned}$$

restricted to the complement of the origin to yield a map

$$\begin{aligned}\wp_0 : S^3 \times \mathbb{R}^+ &\to S^2 \times \mathbb{R}^+, \\ (\mathrm{r}, \varrho) &\mapsto \left(\mathrm{Hopf}(\mathrm{r}), \frac{\varrho^2}{2}\right)\end{aligned}$$

when written in polar coordinates.

Introduce an orthonormal coframe $\{\sigma_1, \sigma_2, \sigma_3\}$ such that the pull-back of the area form of $(S^2, g_{S^2})$ is $4\sigma_1 \wedge \sigma_2$. Since

$$d\sigma_3 = 2\sigma_1 \wedge \sigma_2,$$

we may, after a gauge transformation, take

$$\omega = -\sigma_3 + \wp^* \theta$$

where $\theta$ is a smooth 1-form on a neighborhood of $0 \in \mathbb{R}^3$. Thus

$$\frac{g}{1-z^2} = (1+\varrho^2 F)d\varrho^2 + (1+\varrho^2 \tilde{F})\varrho^2(\sigma_1^2+\sigma_2^2) + (1+\varrho^2 F)^{-1}\varrho^2(\sigma_3 - \wp^*\theta)^2,$$

where $\tilde{F}$ is the smooth function

$$\tilde{F} = 2F \sum_{m=0}^{\infty} \frac{\varrho^{4m}}{(2m+2)!} + 2 \sum_{m=0}^{\infty} \frac{\varrho^{4m+2}}{(2m+4)!}.$$

Since $\wp^*\theta$ vanishes at $0 \in \mathbb{C}^2$, $g$ extends across the origin in $\mathbb{R}^4 = \mathbb{C}^2$, and in fact $\frac{g}{1-z^2}$ agrees with the Euclidean metric $d\varrho^2 + \varrho^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ at the origin. The parallel complex structure $J$ now automatically extends smoothly across the origin of $\mathbb{R}^4$ in such a manner as to be parallel with respect to the metric connection: in fact, its value at 0 may be obtained by parallel transport along any curve passing through 0, this being independent of path because the curvature of $g$ is non-singular.

We now consider the behavior of the metric near $z = -1$. First recall that $V$ is of the form

$$V = 1 + (1-z^2)f$$

for $f$ a smooth function. Thus

$$*dV = f_x dy \wedge dz + f_y dz \wedge dx + \frac{V_z}{y^2} dx \wedge dy$$

is smooth along $z = -1$, $y > 0$, and we can write $*dV = d\varphi$ for $\varphi$ a smooth 1-form defined on a neighborhood of $z = -1$, $y > 0$ in $\mathcal{U} \subset \mathbb{R}^3$. The metric can now be written as

$$g = [1 + f(1-z^2)]\left(\frac{dx^2+dy^2}{y^2} + \frac{dz^2}{1-z^2}\right) + [1+f(1-z^2)]^{-1}(1-z^2)(dt+\varphi)^2$$

near $z = -1$ by choosing a suitable local trivialization for $M_0 \to \mathcal{U} \to p\Gamma - q\Gamma$.

Let us introduce a new variable $q \in [0,1]$ by setting $z = -\sqrt{1-q^2}$. The metric becomes

$$g = (1+q^2 f)\left(\frac{dx^2+dy^2}{y^2} + \frac{dq^2}{1-q^2}\right) + (1+q^2 f)^{-1} q^2 (dt+\varphi)^2 .$$

Notice that, in $(x, y, q)$ coordinates, $\varphi$ has vanishing $dq$ component along $q = 0$. Since $g$ differs from

$$g' = (1+q^2 f)\frac{dx^2+dy^2}{y^2} + (dq^2 + q^2 dt^2)$$

by sums of products of $q\,dt$, $q\,dq$, and $q\varphi$ with smooth coefficients, if we introduce new coordinates $u$ and $v$ by

$$u = q \cos t ,$$

$$v = q \sin t ,$$

we have

$$g = \frac{dx^2+dy^2}{y^2} + du^2 + dv^2 + O(u^2+v^2) .$$

Thus we may adjoin a copy of the half plane $\mathbb{R} \times \mathbb{R}^+$ along $z = -1$ to obtain a larger Riemannian manifold. The complex structure $J$ extends to this manifold, and in fact is given by $dx \mapsto dy$, $du \mapsto dv$ when $u = v = 0$. The adjunction of another half-plane at $z = +1$ arises by the same trick, with $q = \sqrt{1-z^2}$.

Since by construction $*dV$ vanishes when restricted to $z = 0$, our connection $\omega$ is flat over this simply connected real 2-surface. Let $C \subset M$ therefore denote any horizontal lift of this surface. The formula for $J$ then makes $C$ a holomorphic curve, and in fact $x + iy$ is a holomorphic coordinate on $C$.

For each $\alpha \in \Gamma$, we now have a circle's worth of isometric lifts $M \to M$ of the corresponding isometry $\mathscr{U} \to \mathscr{U}$, since, by construction, $V$ is $\Gamma$-invariant; the only ambiguity in choosing our isometry $M \to M$ corresponds to a global gauge transformation of $\pi_0 : M_0 \to \mathscr{U} - p\Gamma - q\Gamma$. Any such lifting will send the holomorphic curve $C$ to another horizontal lift of $z = 0$, and so there is a *unique* lift of $\alpha$ which sends $C$ into itself. With this choice of lifting, we now have an isometric action of $\Gamma$ on $M$. Notice that this action is holomorphic.

We now have set $\Sigma = M/\Gamma$. The map $\mathscr{U} \to \mathscr{H}^2 : (x, y, z) \mapsto x + iy$ then induces a holomorphic map

$$\mu : \Sigma \to S_{\mathrm{g}},$$

where $S_{\mathrm{g}}$ is our Riemann surface of genus g. Since $M \to \mathscr{H}^2$ is a proper map, $\Sigma$ is a compact complex surface, which we will now analyze.

First notice that the line segments

$$\{(a,b)\} \times [c,1] \subset \bar{\mathscr{U}}$$

and

$$\{(a,b)\} \times [-1,-c] \subset \bar{\mathscr{U}}$$

correspond to 2-spheres in $M$, since they are obtained from cylinders $]c,1[ \times S^1$ and $]-1,-c[ \times S^1$ by adjoining two points to obtain compact spaces; inspection of our nuts and bolts procedure even shows that these 2-spheres are smoothly embedded. Moreover, they are complex curves, $\hat{E}_1$ and $\hat{E}_2$, in $M$ because the holomorphic function $x + iy$ is constant on them. Finally, they have self-intersection $-1$, since replacing, say, $\{(a,b)\} \times [c,1]$ with some other line segment from $p = (a,b,c)$ to $z = 1$ results in another 2-sphere which meets our holomorphic curve $\hat{E}_1$ in exactly one point, and the index of this intersection may be shown to be negative by local inspection of the nut construction. These exceptional holomorphic curves $\hat{E}_1$ and $\hat{E}_2$ then project injectively into $\Sigma$ to give exceptional curves $E_1, E_2 \subset \Sigma$ of the first kind.

Let us blow down $\Sigma$ at $E_1$ and $E_2$ to obtain a new compact surface $\check{\Sigma}$. The holomorphic map $\mu : \Sigma \to S_g$ induces a holomorphic projection $\check{\mu} : \check{\Sigma} \to S_g$, and now every fiber is a genus 0 curve, either corresponding to a segment of the form $\{(a', b')\} \times [-1, 1]$, where $(a', b')$ is not in the orbit of $(a, b)$, or to a segment of the form $\{(a,b)\alpha\} \times [-c, c]$ for $\alpha \in \Gamma$. Thus $\check{\mu} : \check{\Sigma} \to S_g$ is a (minimal) ruled surface over $S_g$. Moreover, $\check{\mu}$ has three disjoint holomorphic sections, corresponding to $C/\Gamma$ and to $z = \pm 1$. Hence $\check{\mu}$ is a trivial $\mathbb{P}_1$-bundle, and $\check{\Sigma} \cong S_g \times \mathbb{P}_1$. Thus $\Sigma$ is obtained from $S_g \times \mathbb{P}_1$ by blowing up two points. We have proved the following result:

**Theorem 1.** *Let $\Sigma$ be the complex surface obtained from $S_g \times \mathbb{P}_1$, $g \geqq 2$, by blowing up two distinct points on a fiber $\{\mathrm{pt}\} \times \mathbb{P}_1$. Then $\Sigma$ admits a Kähler metric of zero scalar curvature.*

In fact, we can even say what the Kähler class is, since the exceptional divisors $E_1$ and $E_2$ have area $2\pi(1-c)$, a fiber of $\mu$ has area $4\pi$, and the proper transform of the surface $S_g$ given by $z = -1$ has area $4\pi(g-1)$, so that $C/\Gamma = S_g \times \{\mathrm{pt}\}$ has area $4\pi(g-1) + 2\pi(1-c)$.

Let us close this section by noticing that we could have instead chosen $V$ to be given by $1 + G_{p_1\Gamma} + G_{q_1\Gamma} + \ldots + G_{p_m\Gamma} + G_{q_m\Gamma}$, where the points $\{p_j, q_j\}$ are symmetrically placed and have distinct orbits under $\Gamma$. This allows us to endow the blow up of $S_g \times \mathbb{P}_1$ at a collection of the form $\{m \text{ points}\} \times \{2 \text{ points}\}$ with a scalar flat Kähler metric; in fact, some of these points may even be allowed to coincide, since there is no reason why we may not take several of the (distinct) $p_j$'s to have the same $(x, y)$-coordinates. In other words:

**Theorem 2.** *There are zero scalar curvature Kähler metrics on the blow-up of $S_g \times \mathbb{P}_1$ at any even number of points, at least when the points are arranged in a sufficiently symmetrical manner.*

Note that, by contrast, the blow-up of $S_g \times \mathbb{P}_1$ at *one* point cannot admit a constant scalar curvature Kähler metric, since its automorphism group is not reductive [15].

## § 5. Other ruled surfaces

We conclude this article by noticing that the same technique also yields scalar-flat Kähler metrics on blow-ups of certain ruled surfaces which are non-trivial $\mathbb{P}_1$-bundles.

Let $p$ and $q$ be two points of the half-plane $z = 0$, $y > 0$; we will assume for simplicity that they are distinct elements of the same fundamental domain $\mathscr{R}$ of $\Gamma$, and let $\gamma$ be an arc from $p$ to $q$ which is contained in the interior of $\mathscr{R}$. Again we take

$$V = 1 + G_{p\Gamma} + G_{q\Gamma},$$

and consider the complex surface $M$ obtained by completing the circle bundle

$$\pi_0 : M_0 \to \mathscr{U} - p\Gamma - q\Gamma$$

by the "nuts and bolts" procedure. We would now like to make $\Gamma$ act on $M$.

To do this, notice that the connection $\omega$ on $\pi_0$ is again flat when restricted to $z = 0$, since again $V(x, y, z) = V(x, y, -z)$. Nonetheless, this restricted connection is not *a priori* trivial, since the complement of $p\Gamma \cup q\Gamma$ in $z = 0$ is not simply-connected.

In fact, the holonomy of this flat connection is exactly $\mathbb{Z}_2$, and parallel transport around a loop just counts, modulo 2, the number of points of $p\Gamma \cup q\Gamma$ enclosed by the loop. To see this, notice that such a loop in $z = 0$ bounds a surface $S_0$ in $z \geqq 0$, and the holonomy transformation around the loop is thus given by

$$e^{i\int_{S_0} d\omega} = e^{i\int_{S_0} *dV}.$$

But if $S$ is the closed surface consisting of $S_0$ and its reflection in $z$, the fact that $V$ is even in $z$ implies that

$$\int_{S_0} * dV = \frac{1}{2} \int_S * dV = -\pi m$$

where $m$ is the number of points of $p\Gamma \cup q\Gamma$ enclosed by the surface $S$ in 3-space, or equivalently the number of points of $p\Gamma \cup q\Gamma$ enclosed by the given loop $\partial S_0$ in the plane $z = 0$.

We conclude that the connection $\omega$ is trivial on the complement of $\gamma\Gamma$ in $z = 0$. Let $C_0$ denote a horizontal lift of this region $\mathscr{H}^2 - \gamma\Gamma$, which is a holomorphic curve in $M$. We then lift the action of $\Gamma$ to $M$ by requiring that it stabilize $C_0$. Set $\Sigma = M/\Gamma$.

$\Sigma$ is again not a minimal surface. Indeed, if $p = (a, b, 0)$ and $q = (a', b', 0)$, the inverse images of $\{(a, b)\} \times [0, 1]$ and $\{(a', b')\} \times [0, 1]$ are rational curves $\hat{E}_1$ and $\hat{E}_2$ in $M$ with self intersection $-1$; their images $E_1$ and $E_2$ in $\Sigma$ are then disjoint exceptional divisors of the first kind. Let $\check{\Sigma}$ denote the surface obtained by blowing these curves down.

As in the previously considered case, $\Sigma$ comes equipped with a holomorphic projection

$$\mu : \Sigma \to S_g$$

which induces a projection

$$\check{\mu} : \check{\Sigma} \to S_g;$$

again, $\check{\mu}$ is a $\mathbb{P}_1$ bundle over $S_g$.

This time, however, the bundle is not trivial. In fact, if we continue $C_0$ as a horizontal submanifold of $M_0$ and then take its closure in $M$, the result is a complex curve $C$ which is a 2-fold branched cover of the half-plane $z = 0$, $y > 0$, with branch locus $p\Gamma \cup q\Gamma$. Descending to the quotient $\Sigma = M/\Gamma$, the curve $C/\Gamma$ is a 2-fold branched cover of $S_g$ via the map $\mu : \Sigma \to S_g$. If we then consider its image in $\check{\Sigma}$, we find that it meets the surface corresponding to $z = 1$ in exactly the two point corresponding to $E_1$ and $E_2$. Thus

$$\check{\Sigma} \cong \mathbb{P}(\mathcal{L} \oplus \mathcal{O})$$

where $\mathcal{L} \to S_g$ is a line bundle such that $\mathcal{L}^{\otimes 2}$ is the divisor of two distinct but arbitrary points of $S_g$. Notice that changing the $\Gamma$-action by an element of $H^1(S_g, \mathbb{Z}_2)$ alters our square-root $\mathcal{L}$ of this divisor in the obvious manner, and we therefore sweep out all choices of square-root in this way. Thus we have proved the following:

**Theorem 3.** *Let $r_1$ and $r_2$ be arbitrary points of a Riemann surface $S_g$ of genus $g \geqq 2$. Let $\mathcal{L}$ be a square-root of the divisor line bundle $[r_1] + [r_2]$, and let*

$$\check{\Sigma} = \mathbb{P}(\mathcal{L} \oplus \mathcal{O})$$

*be the associated $\mathbb{P}_1$ bundle over $S_g$. Blow up $\check{\Sigma}$ at the points corresponding to the zero vectors $0_{r_1}$ and $0_{r_2}$ of the fibers of $\mathcal{L}$ over $r_1$ and $r_2$. Then the resulting manifold $\Sigma$ admits a scalar-flat Kähler metric.*

Note that $\check{\Sigma}$ is not even homeomorphic to the product $\mathbb{P}_1 \times S_g$, so this result is quite different from Theorem 1.

Finally, let us consider a combination of chosen points, some as in Theorem 3, some as in Theorem 1, and again take

$$V = 1 + \Sigma(G_{p_j\Gamma} + G_{q_j\Gamma}).$$

The result is the following:

**Theorem 4.** *Let $\mathcal{L} \to S_g$ be the square-root of the divisor of an even number $2m$ of points. If $\mathbb{P}(\mathcal{L} \oplus \mathcal{O})$ is blown up at $2l$ points, $l \geqq m$, in sufficiently special position, then the resulting surface $\Sigma$ admits a scalar-flat Kähler metric.*

Here "sufficiently special" means that the zero vectors of the divisor points are to be blown up, and the other points are constrained to lie on the "zero" and "infinity" sections of $\mathbb{P}(\mathcal{L} \oplus \mathcal{O})$, paired up in the obvious manner.

So far, we have proceeded by first constructing the universal cover $M$ of $\Sigma$, and then taking the quotient by a $\Gamma$-action, which must also be constructed. However, there is a more robust alternative available. We may instead think of $\mathscr{U}/\Gamma = S_g \times ]-1,1[$ as a hyperbolic 3-manifold, and construct a complex surface $\Sigma \to S_g$ out of a harmonic function

$$V = 1 + \Sigma G_{p_j\Gamma}$$

on $(S_g \times ]-1,1[) - \{[p_j]\}$ provided that $\frac{1}{2\pi} * dV$ defines an integral class in deRham cohomology. This condition will be satisfied iff

$$\frac{1}{2\pi} \int_{S_g \times \{t\}} * dV \in \mathbb{Z}$$

for some (and hence any) $t \in ]-1,1[$ distinct from all the $z$-coordinates $c_j$ of the given points $[p_j] \in S_g \times ]-1,1[$. Fortunately, integrals of this type can be evaluated explicitly, using the following observation:

**Lemma 2.** *For a point* $[p] = p\Gamma \in S_g \times ]-1,1[$ *with z-coordinate c,*

$$\int_{S_g \times \{1-\varepsilon\}} * dG_{p\Gamma} = -\pi(1+c),$$

*provided that* $0 < \varepsilon < 1 - c$.

*Proof.* Letting $\mathscr{R} \subset \mathscr{H}^2 \times \{1-\varepsilon\} \subset \mathscr{U}$ denote a fundamental domain in

$$\mathscr{H}^2 \times \{1-\varepsilon\}$$

for the $\Gamma$-action, we begin by observing that

$$\begin{aligned} \int_{S_g \times \{1-\varepsilon\}} * dG_{p\Gamma} &= \int_{\mathscr{R}} * d \sum_{\alpha \in \Gamma} G_{p\alpha} \\ &= \sum_{\alpha \in \Gamma} \int_{\mathscr{R}\alpha^{-1}} * dG_p \\ &= \int_{\mathscr{H}^2 \times \{1-\varepsilon\}} * dG_p . \end{aligned}$$

In order to compute the latter integral, note that, in exponential polar coordinates about $p \in \mathscr{H}^3$, $* dG_p$ is minus one-half the area form of the 2-sphere, so the above integral essentially gives the measure of the set of geodesics through $p$ which pass through the surface $\mathscr{H}^2 \times \{1-\varepsilon\}$. The latter is most easily determined by passing to the upper half-plane model via the coordinate change $(r, s, t) = (x, yz, y\sqrt{1-z^2})$ mentioned in § 2. Let the coordinates of $p$ be $(r, s, t) = (\alpha, \beta, \gamma)$, so that $c = \frac{\beta}{\sqrt{\beta^2 + \gamma^2}}$. The integral we wish to evaluate is then minus one-half the area of the half-plane $t = 0$, $s > 0$ with respect to the area form induced by the exponential map at $p$, since, assuming that $0 < \varepsilon < 1 - c$, a directed hyperbolic geodesic through $p$ terminates on $t=0$, $s>0$ iff it first passes through the half-plane $r=\frac{\sqrt{\varepsilon(2-\varepsilon)}}{1-\varepsilon}s$, $t>0$. Now the exponential map from the unit tangent sphere at $p$ to the sphere at infinity of $\mathscr{H}^3$ is a conformal map (as is made most clear by considering the Poincaré ball model of $\mathscr{H}^3$, with $p$ at the origin), and, in the upper-half-space model, sends the north pole to infinity, the south pole to $(r,s,t)=(\alpha,\beta,0)$, and the equator to $r^2+s^2=\gamma^2$, $t=0$. Notice, however, that these properties are shared by stereographic projection to the plane $t=0$ from a unit 2-sphere in $\mathbb{R}^3$ with north pole $p$. These two maps are therefore identical. We must therefore calculate minus one-half the area of the region of a unit 2-sphere below a plane through the north pole, the normal vector of which makes an angle of $\phi:=\cos^{-1}\frac{\beta}{\sqrt{\beta^2+\gamma^2}}$ with the vertical. The latter region, however, is just the complement of a spherical disk of angular radius $\phi$. Hence

$$\begin{aligned}\int_{\mathscr{H}^2\times\{1-\varepsilon\}} * dG_p &= -\frac{1}{2}[4\pi-2\pi(1-\cos\phi)]\\ &= -\pi(1+\cos\phi)\\ &= -\pi(1+c). \qquad \text{QED}\end{aligned}$$

We thus find a much weaker set of conditions for carrying out our construction: it is sufficient that the set $\{c_j\}$ of $z$-coordinates for our centers $\{[p_j]\}\subset S_{\mathrm{g}}\times]-1,1[$ satisfy

(1) $$\sum_{j=1}^{n} c_j \in \mathbb{Z};$$

and

(2) $$\sum_{j=1}^{n} c_j \equiv n \pmod 2.$$

This can be achieved (with $-1<c_j<1$) for any $n\geqq 2$. Thus we obtain:

**Theorem 5.** *For all natural numbers $n\geqq 2$, there are scalar-flat Kähler surfaces $\Sigma$ obtained from minimal ruled surfaces $\check{\Sigma}\to S_{\mathrm{g}}$, $\mathrm{g}\geqq 2$, by blowing up exactly $n$ points.*

In contrast to our previous results, however, we do not (at present) have a systematic method for determining which minimal ruled surfaces $\check{\Sigma}$ correspond to given configurations of points.[2]) However, the minimal models of surfaces arising from our construction will always be of the form

$$\check{\Sigma}=\mathbb{P}(\mathscr{L}\oplus\mathscr{O})$$

for some holomorphic line bundle $\mathscr{L}\to S_{\mathrm{g}}$, since $\check{\Sigma}\to S_{\mathrm{g}}$ automatically has two sections, corresponding to $z=\pm 1$; moreover, $\Sigma$ is obtained from $\check{\Sigma}$ by blowing up points along the

[2]) **Note added in proof.** Indeed, $\check{\Sigma}$ depends on an extra piece of data, namely a flat circle bundle over $S_{\mathrm{g}}$. For this reason, $\mathscr{L}$ can be taken to be *any* line bundle of the appropriate degree.

'zero' and 'infinity' sections of $\mathbb{P}(\mathscr{L} \oplus \mathcal{O})$. In order to specify our minimal model $\check{\Sigma}$ uniquely, we may choose to require that all the blow-ups occur on the zero section. The degree of $\mathscr{L}$ will then equal $\frac{1}{2}\left(n + \sum_{j=1}^{n} c_j\right)$.

In closing, let us remark that the same techniques explored here can also be applied to other Kleinian groups. For example, if we replace our compact Riemann surface $S_g$ with a compact Riemann surface minus a collection of disks, the resulting self-dual manifolds compactify to yield explicit anti-self-dual metrics on $m(S^1 \times S^3) \# n\overline{\mathbb{C}\mathbb{P}_2}$. Details of a number of such constructions are currently being worked out by Jongsu Kim.

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Department of Mathematics, State University of New York, Stony Brook, NY 11793

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