

Desingularizations of
Conformally Kähler,
Einstein Orbifolds

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Differential Geometry Seminar
University of California, Berkeley
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Most recent results:

Joint work with

Joint work with

Tristan Ozuch

Joint work with

Tristan Ozuch

Massachusetts Institute of Technology

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Massachusetts Institute of Technology

e-print:

arXiv:2601.19215 [math.DG]

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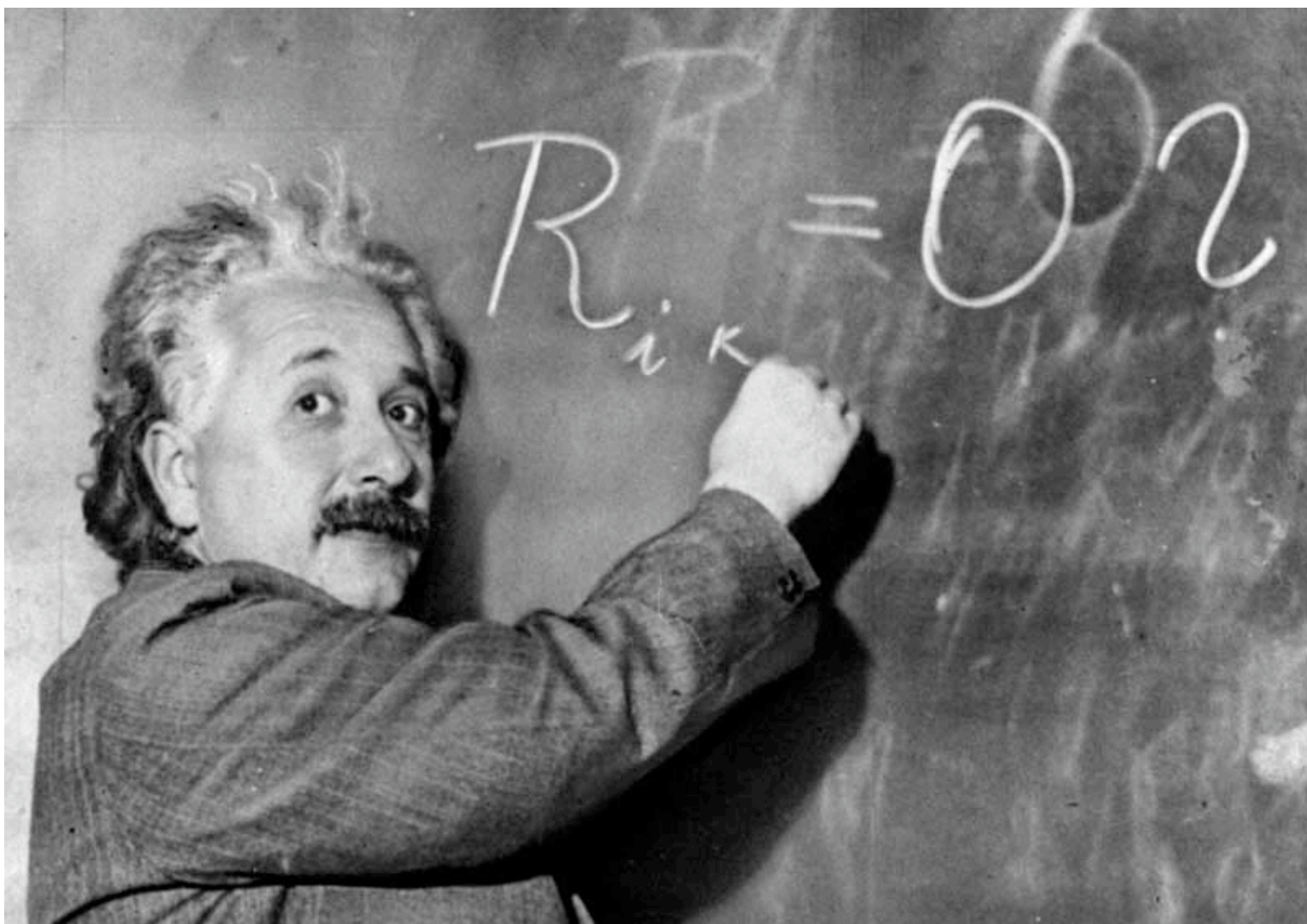
“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow

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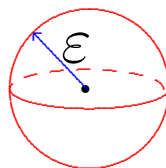
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Similarly in dimension $n = 4k$, $k \geq 2$.

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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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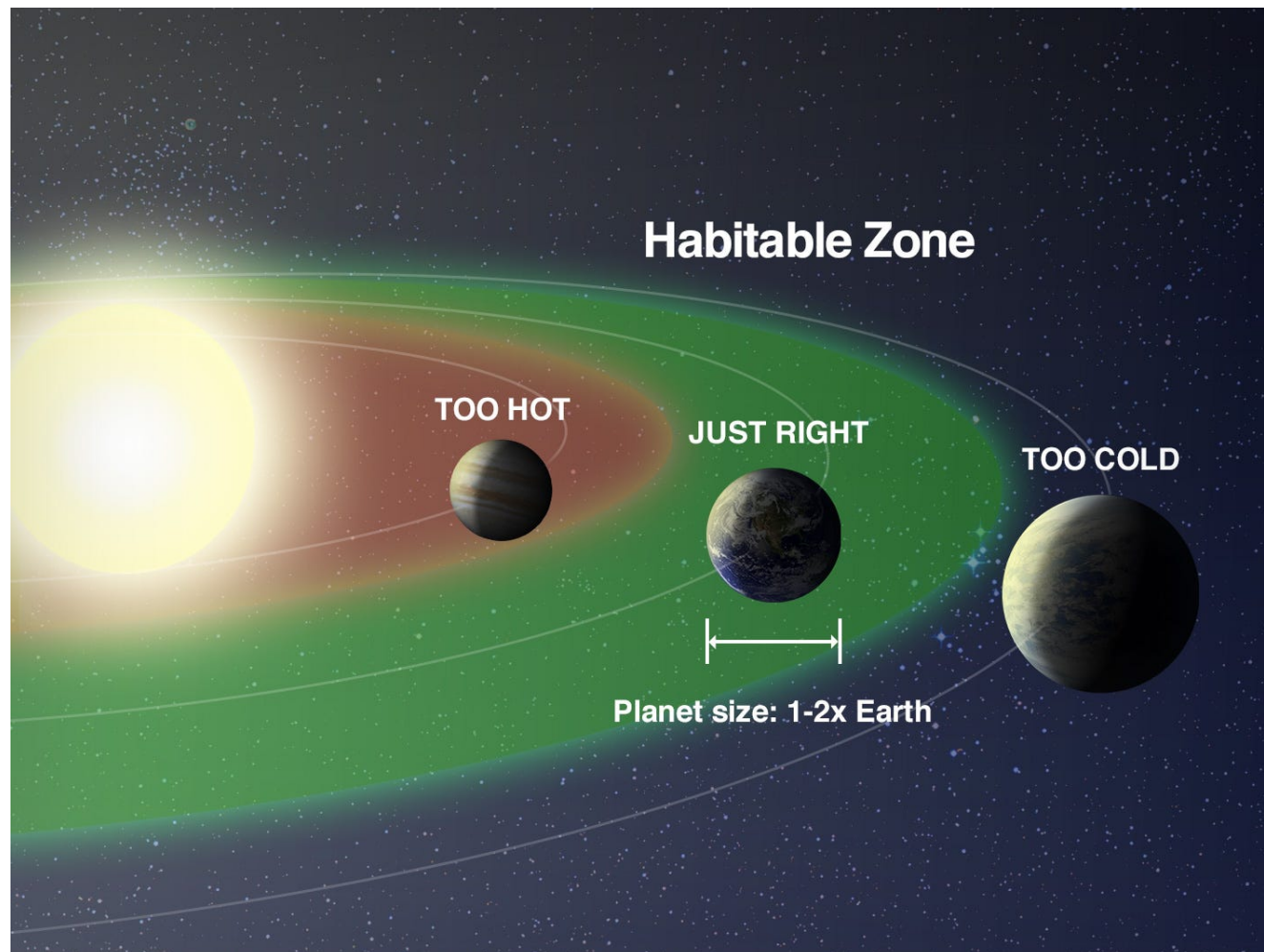
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Λ^+ self-dual 2-forms.

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Which 4-manifolds admit Einstein metrics?

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$$\omega = dx \wedge dy + dz \wedge dt$$

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)? What if we also require $\lambda > 0$?*

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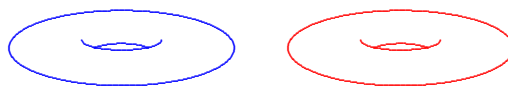
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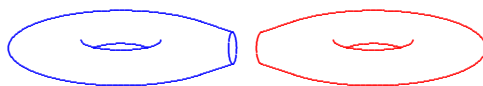
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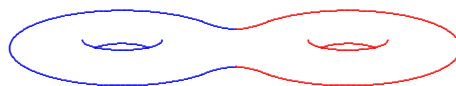
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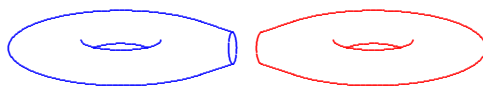
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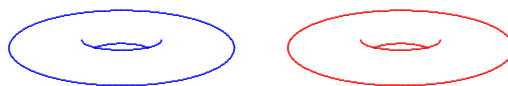
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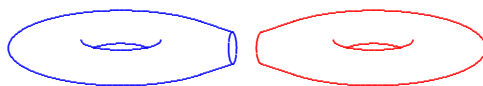
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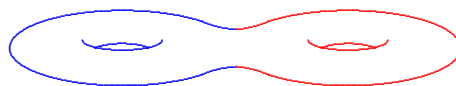
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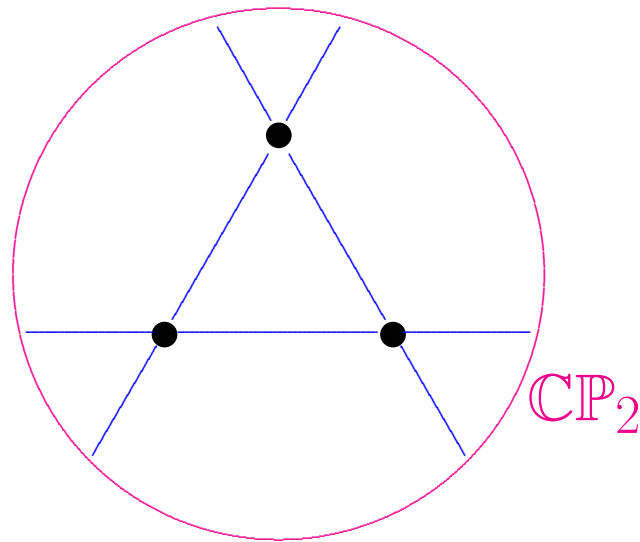
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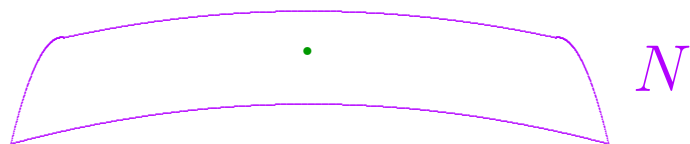
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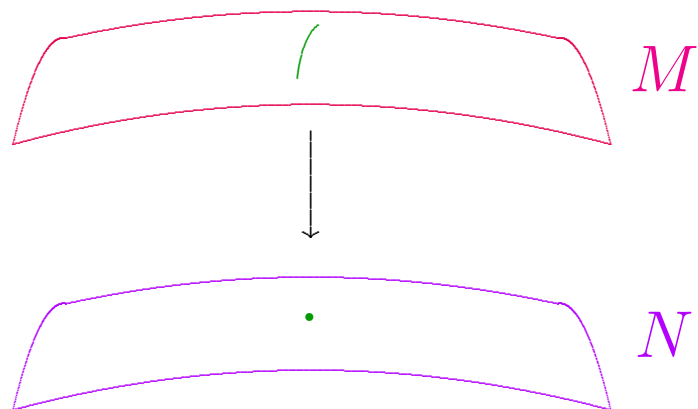
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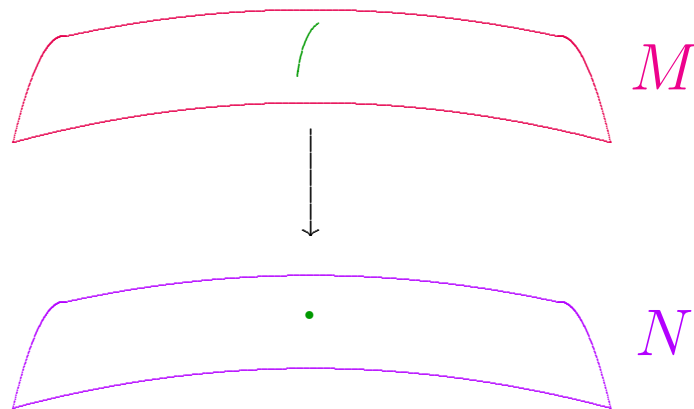
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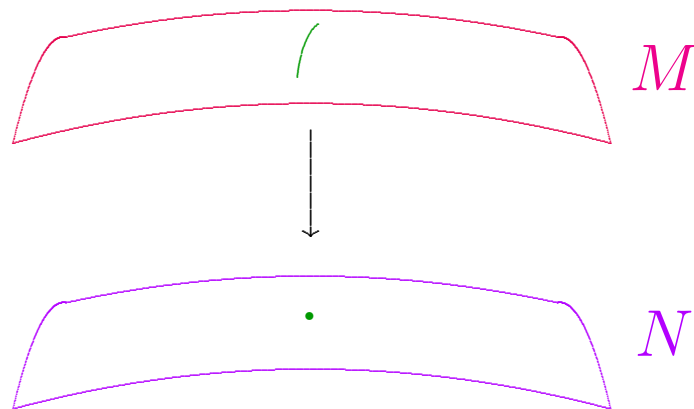


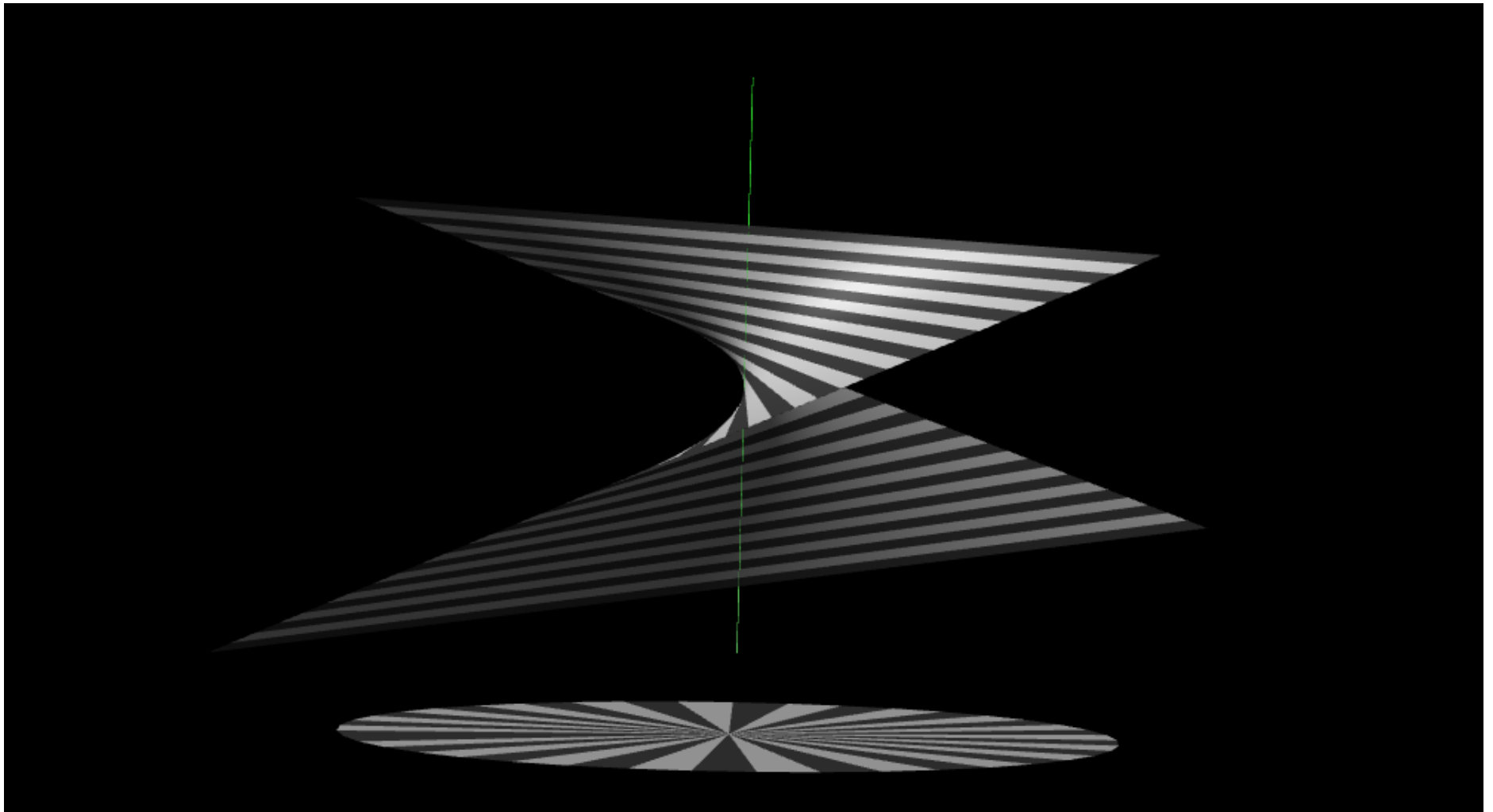
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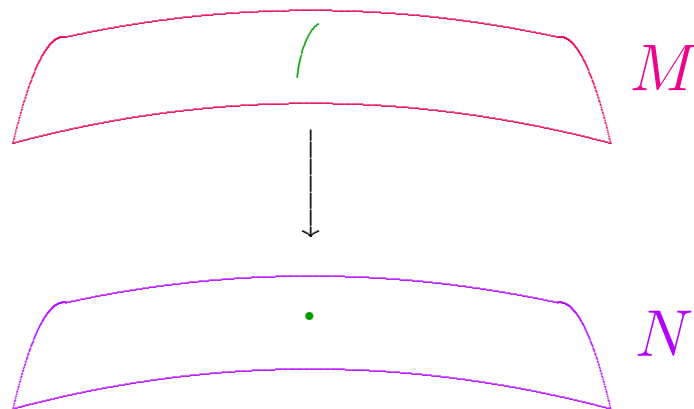


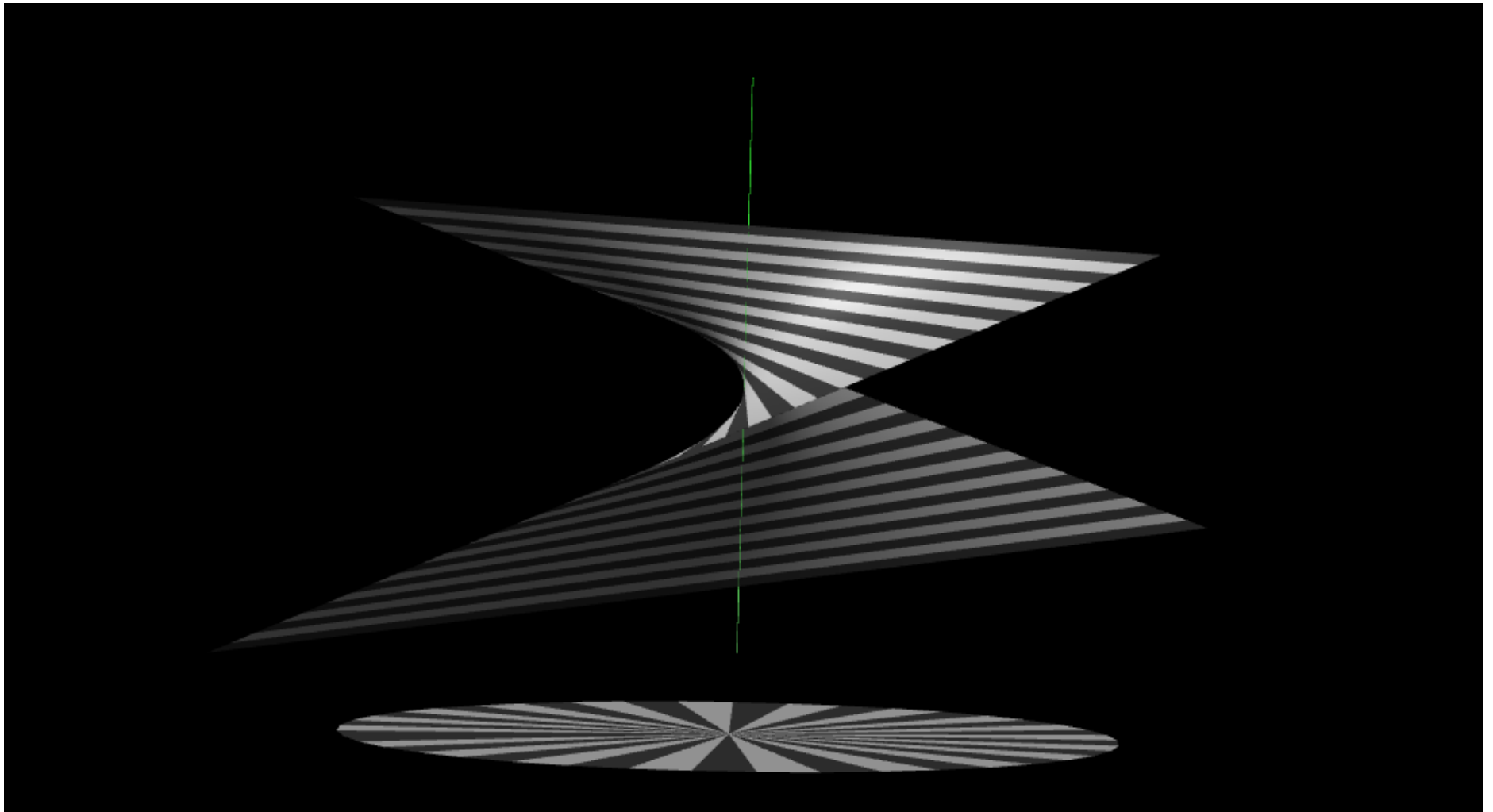
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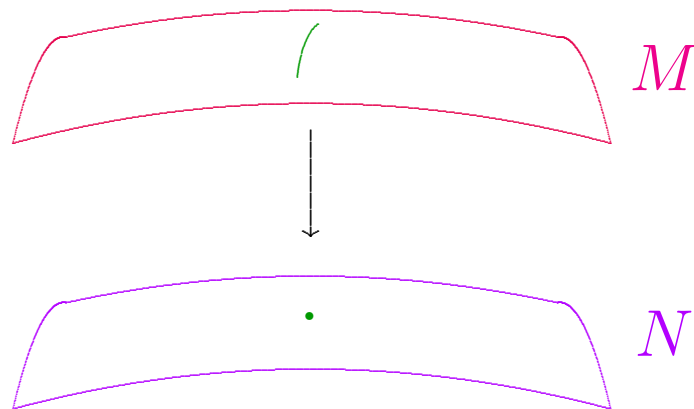


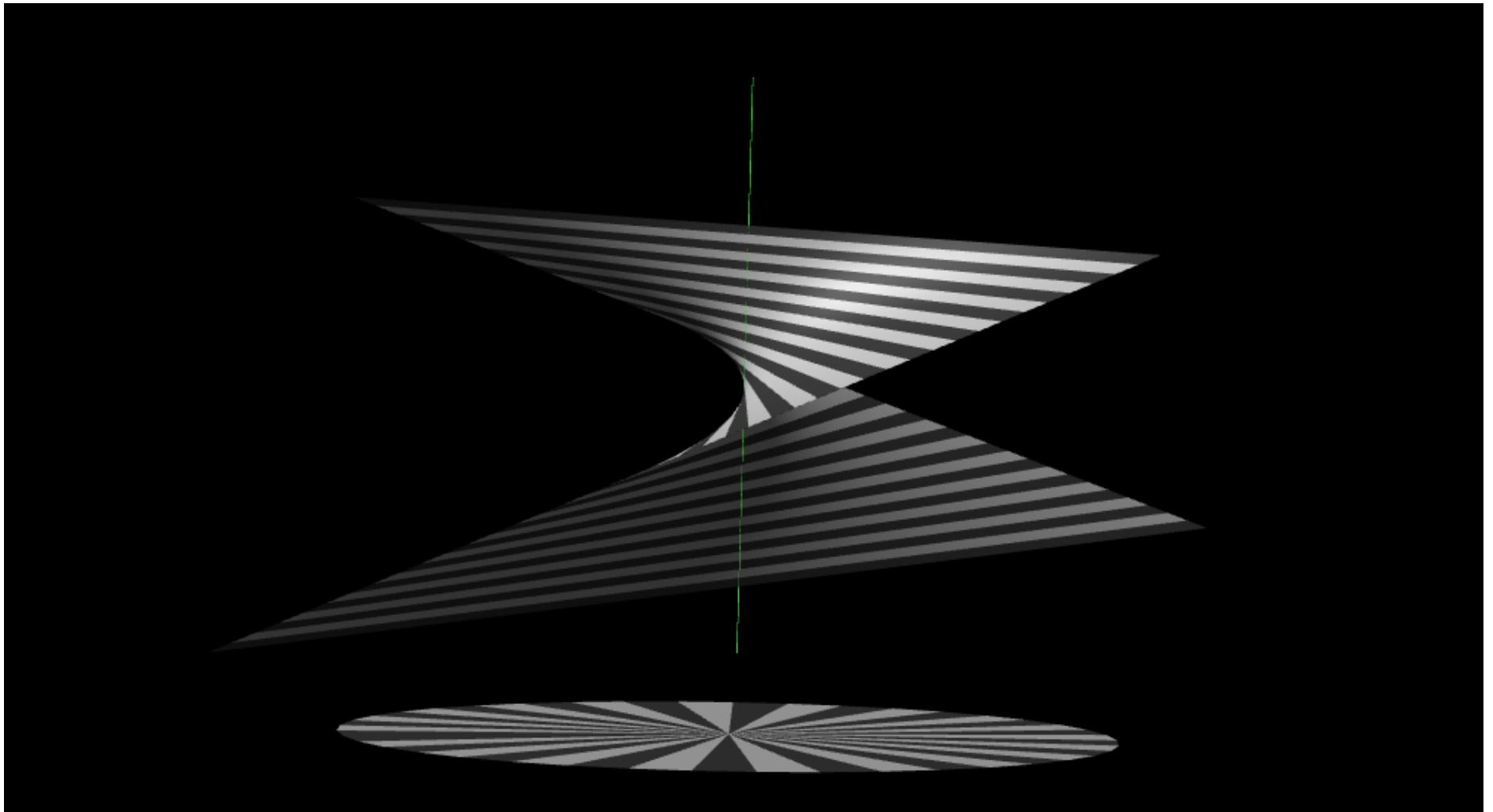
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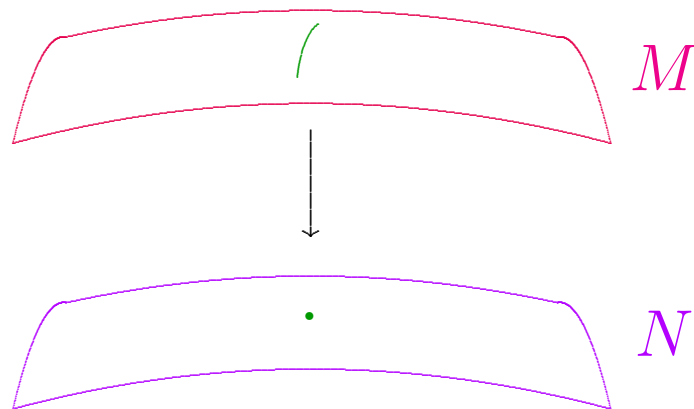


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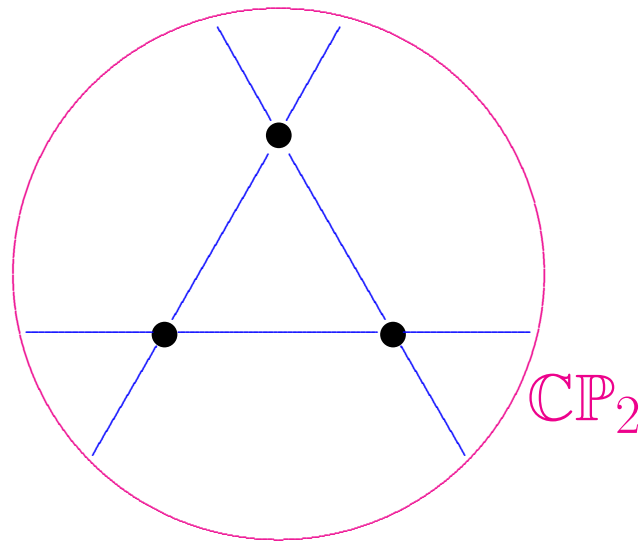


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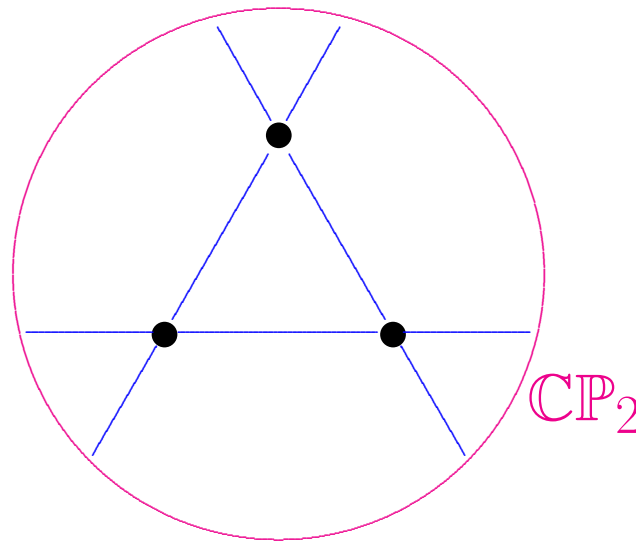


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in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



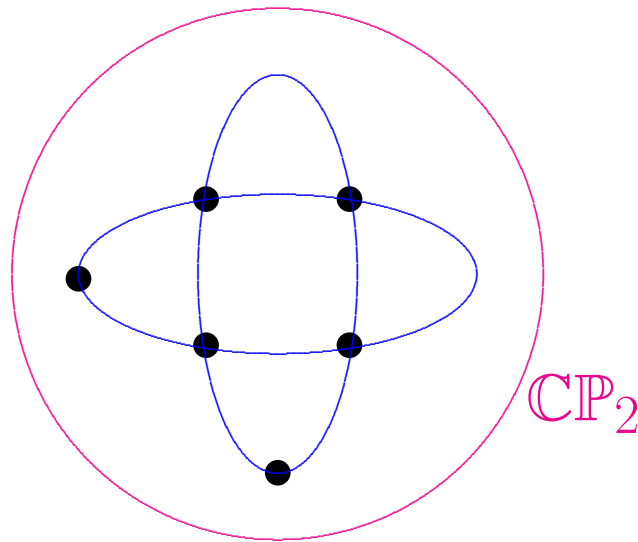
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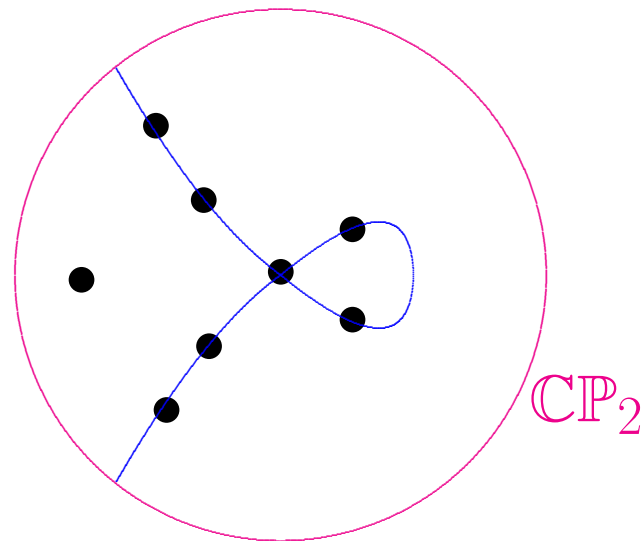
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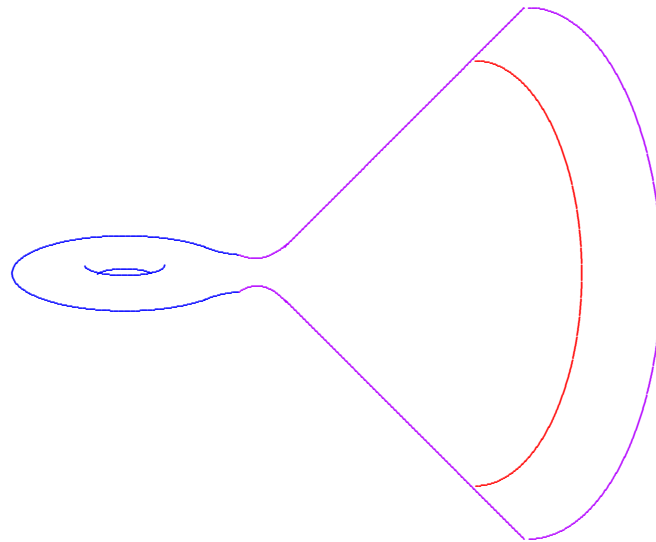
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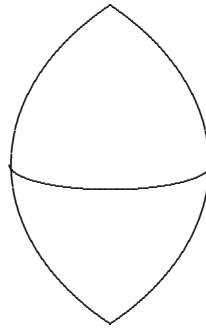
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Goal: Show that this doesn't change anything!

Technical Hitch!

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We avoid this question by means of a definition!

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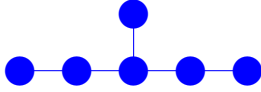
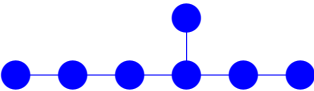
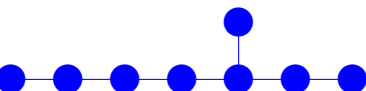
- *Finite subgroups $\Gamma < SU(2)$.*

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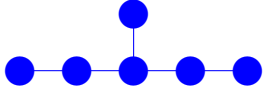
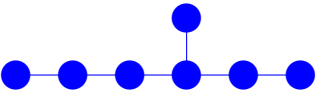
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A_k 	<i>Cyclic</i>	
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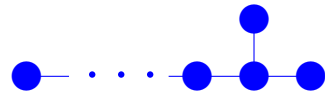
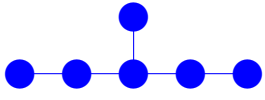
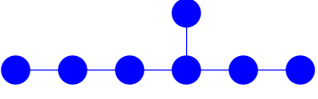
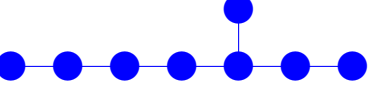
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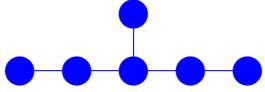
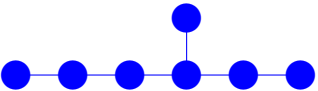
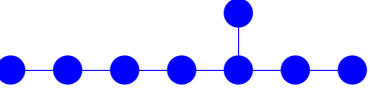
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Standard shorthand: $\frac{1}{\ell m^2}(1, \ell mn - 1)$.

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Proposition. *Either there are infinitely many topological types of compact **K-E** 4-dimensional orbifolds with only isolated singularities that cannot arise as Gromov-Hausdorff limits of smooth Einstein manifolds,*

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

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Proposition. *Either there are infinitely many topological types of compact **K-E** 4-dimensional orbifolds with only isolated singularities that cannot arise as Gromov-Hausdorff limits of smooth Einstein manifolds, or else there are infinitely many Ricci-flat ALE 4-manifolds that remain to be discovered.*

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These examples form part of a systematic picture...

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Classified Gromov-Hausdorff limits of Del Pezzos.

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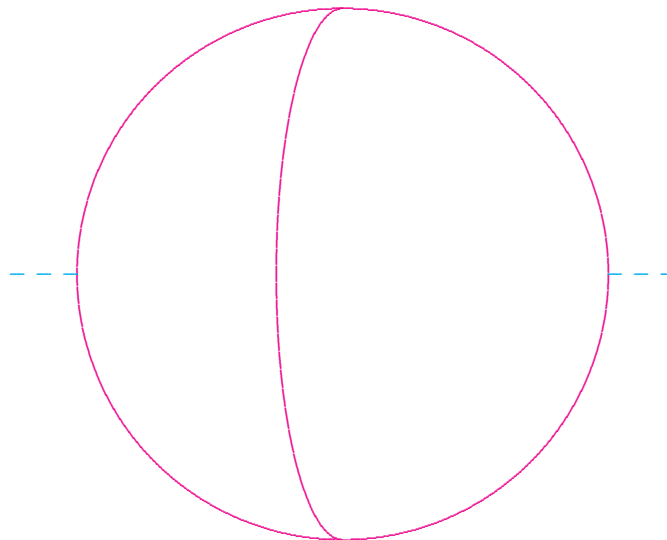
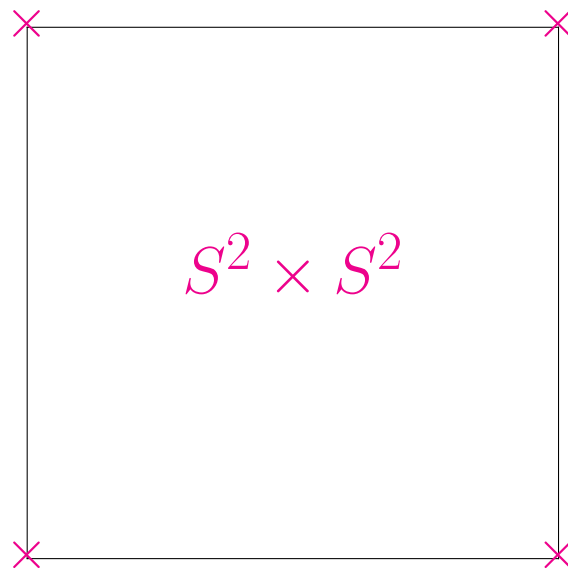
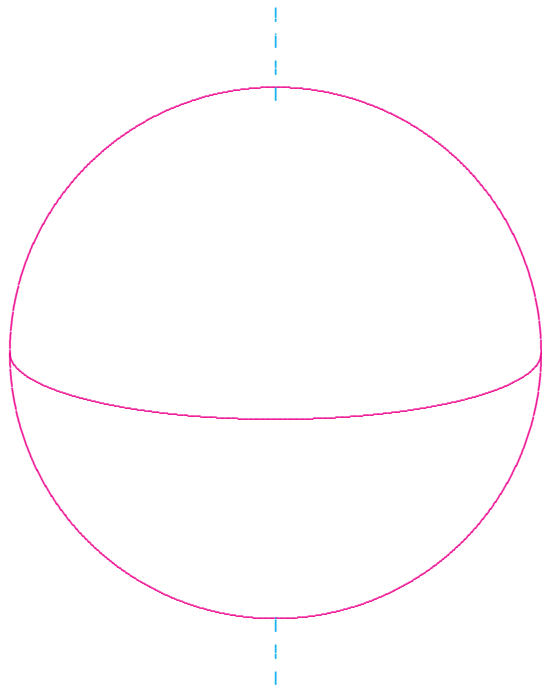
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- $\mathbb{CP}_2 \#^8 \overline{\mathbb{CP}}_2$: only (!) $A_1, \dots, A_{10}, D_4, \frac{1}{4}(1, 1), \frac{1}{8}(1, 3),$ or $\frac{1}{9}(1, 2)$ singularities arise.

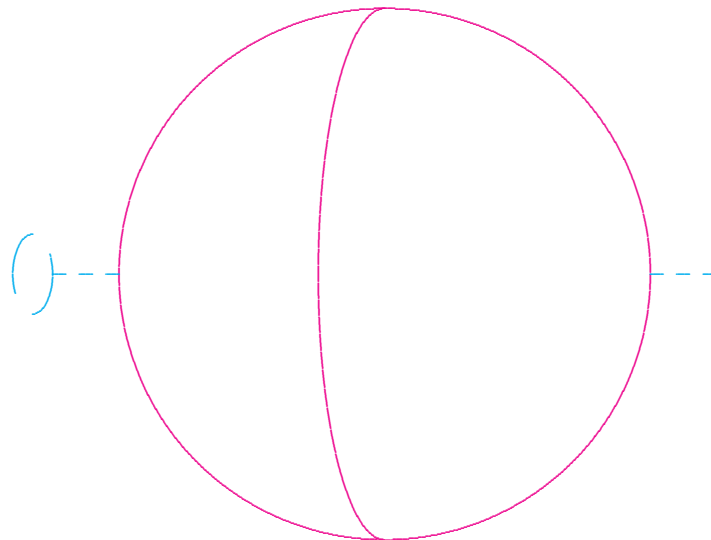
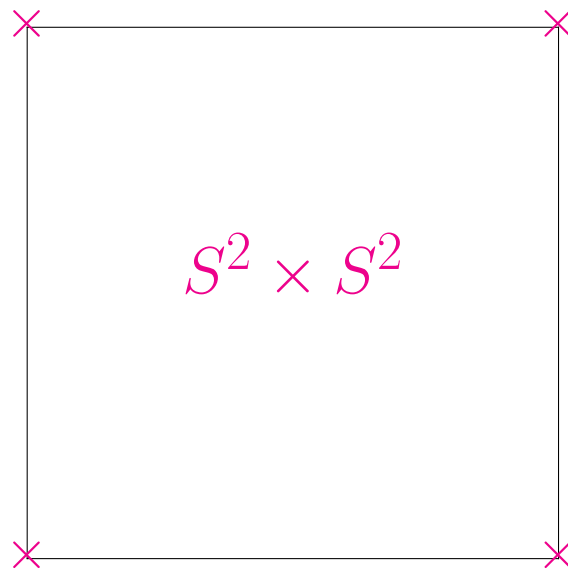
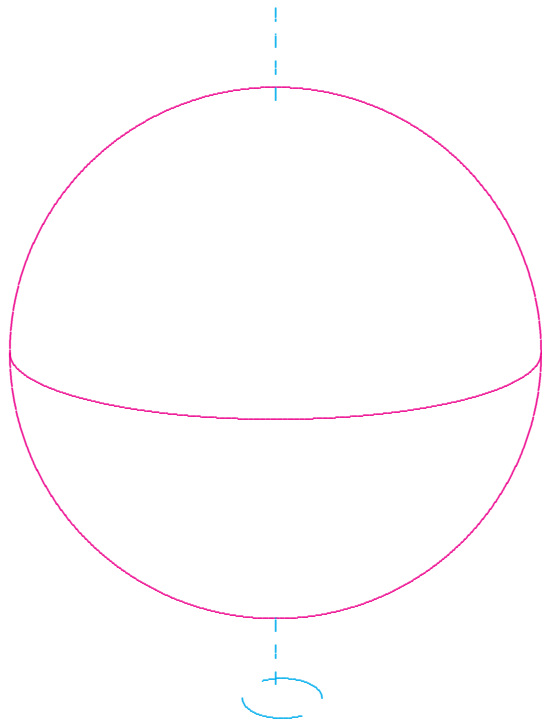
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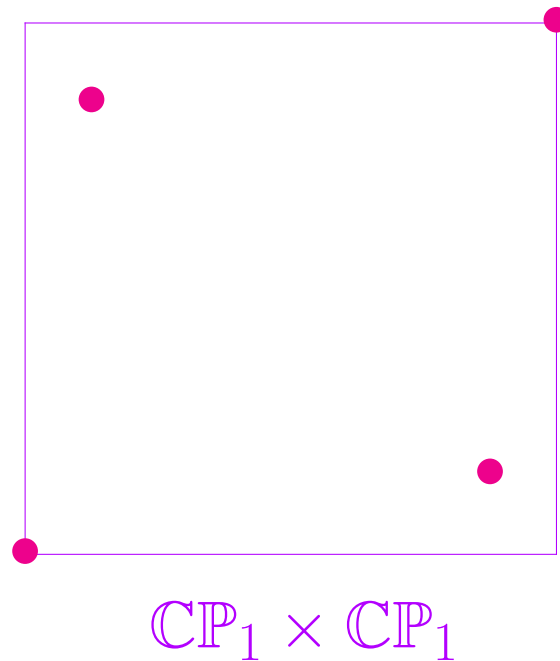
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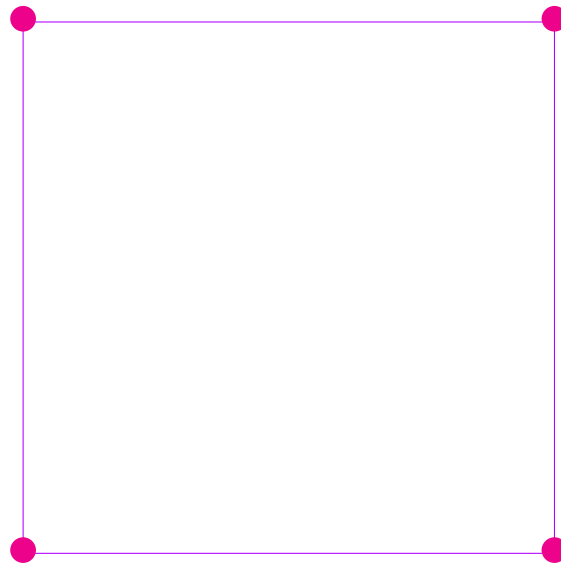
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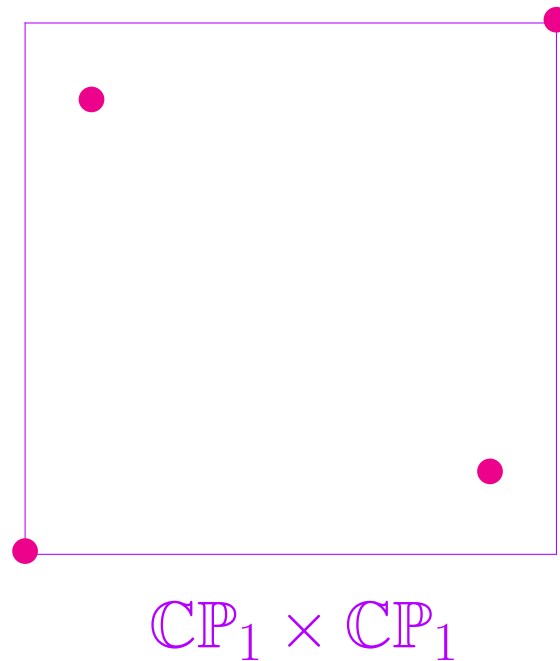


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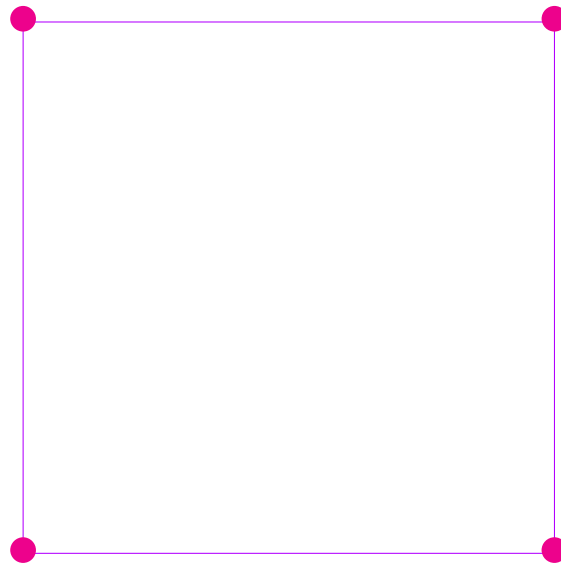
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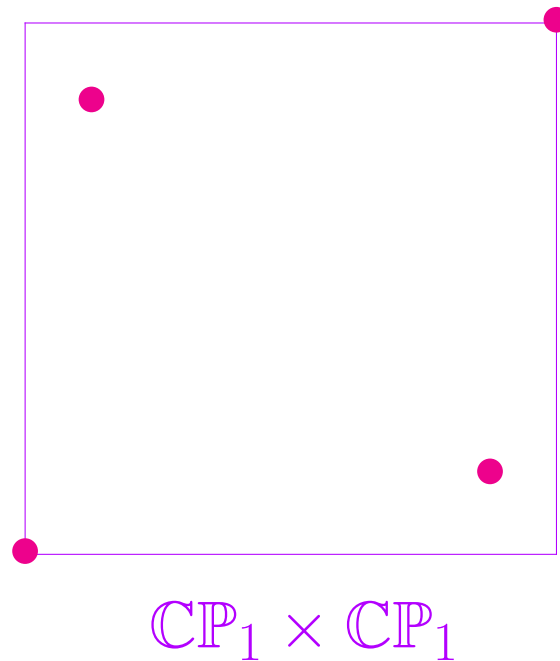


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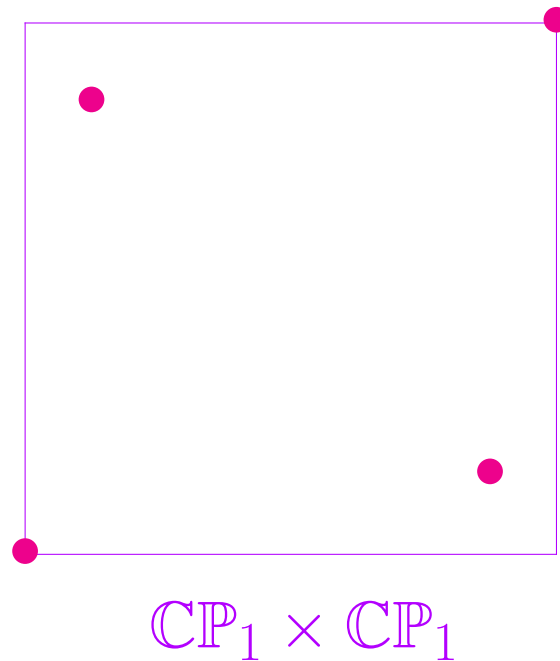
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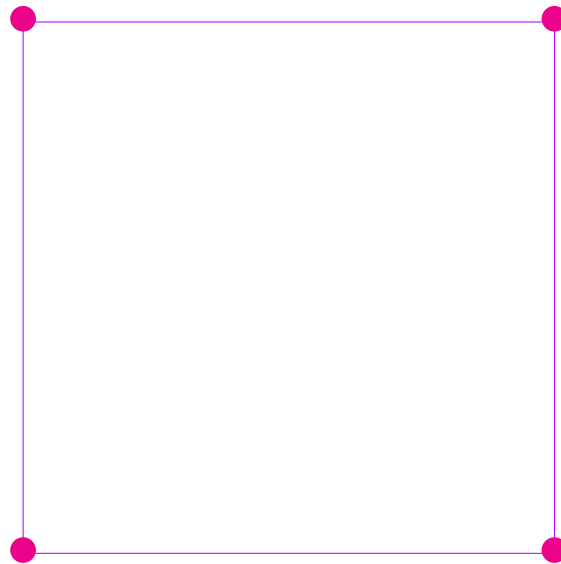
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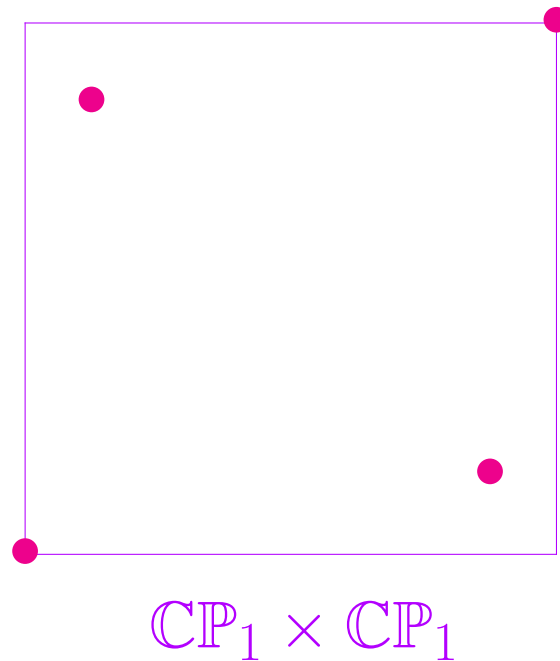


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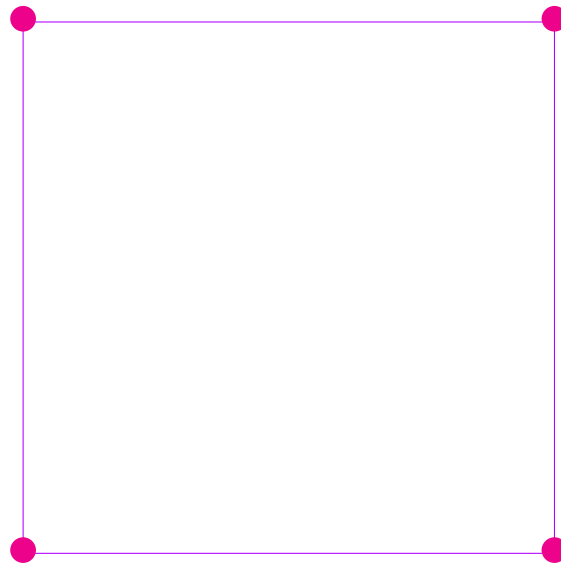
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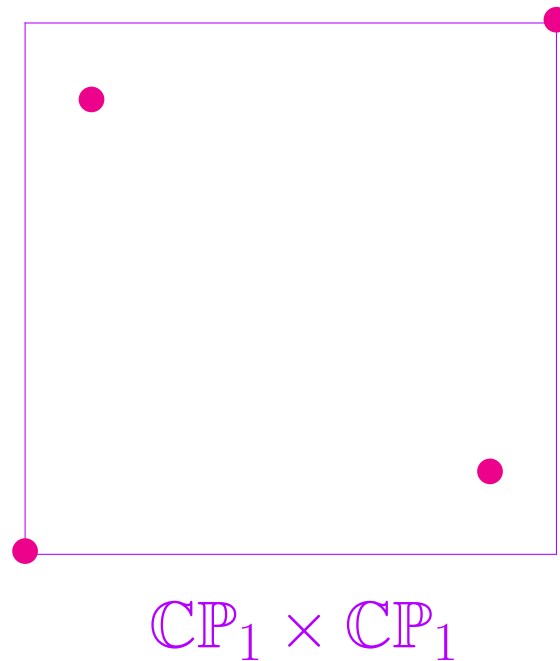


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But what about limits of general Einstein metrics?

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By the Riemannian Goldberg-Sachs Theorem,

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the orbifold Einstein metric g_∞ is **conformally Kähler**.

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A more transparent proof was then given in L '21.

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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And hence that they are actually conformally Kähler!

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