Einstein Metrics,

Curvature Functionals, and

Conformally Kähler Geometry

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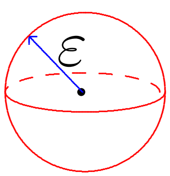
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Try to find Einstein metrics by minimizing?

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is realized by an Einstein metric g_j with $\lambda < 0$.

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On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.

More Modest Question. *If (M^4, J) is a compact complex surface, when does M^4 admit an Einstein metric g (unrelated to J)?*

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Kähler if the 2-form

$$\omega = g(J\cdot, \cdot)$$

is closed:

$$d\omega = 0.$$

But we do not assume this!

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Only two metrics arise in non-Kähler case!

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In other words,

$$g = f\tilde{g}$$

$\exists$ Kähler metric $\tilde{g}$, smooth function $f : M \to \mathbb{R}^+$.

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But $S^3 \times S^3$ has no Kähler metric because $H^2 = 0$.

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But also natural and interesting to consider

$$g \longmapsto \int_M |r|_g^2 d\mu_g$$

or

$$g \longmapsto \int_M |\mathcal{R}|_g^2 d\mu_g$$

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However, these are not independent!

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$\therefore$ Einstein metrics critical $\forall$ quadratic functionals!

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Natural Question. *When does Einstein metric* g *on* 4*-manifold* M *minimize one or both of these functionals?*

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Critical in $[g] \iff s =$ constant.

Minimizer in $[g] \iff g$ is "Yamabe metric."

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Einstein metrics with $\lambda > 0$ never minimize $\int_M s^2 d\mu$!

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Proof 4-dimensional in details, but not philosophy.

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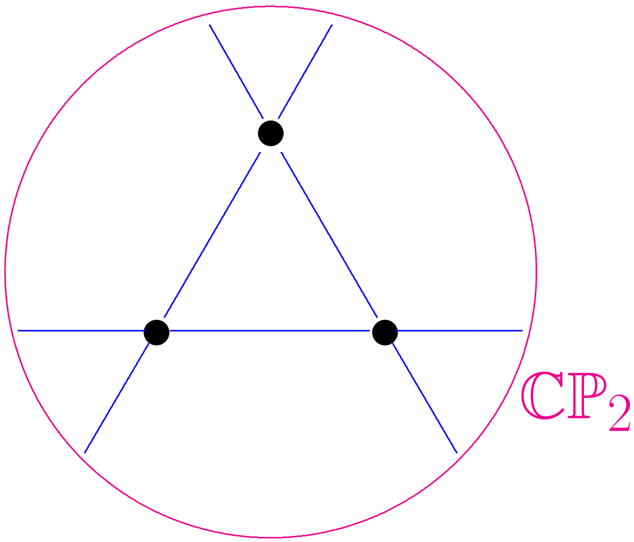
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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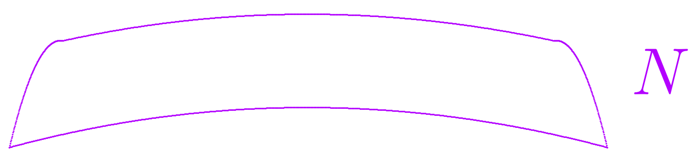

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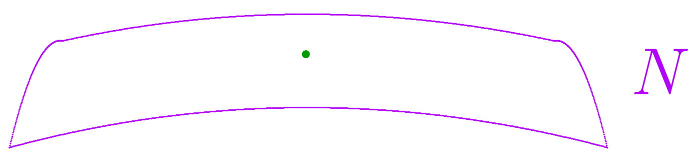
If N is a complex surface, may replace $p \in N$



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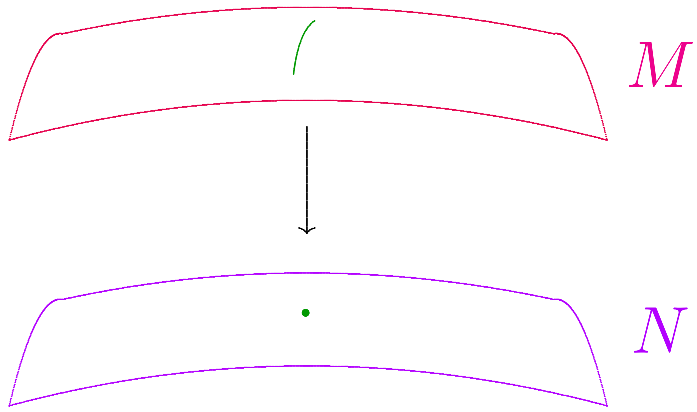
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$

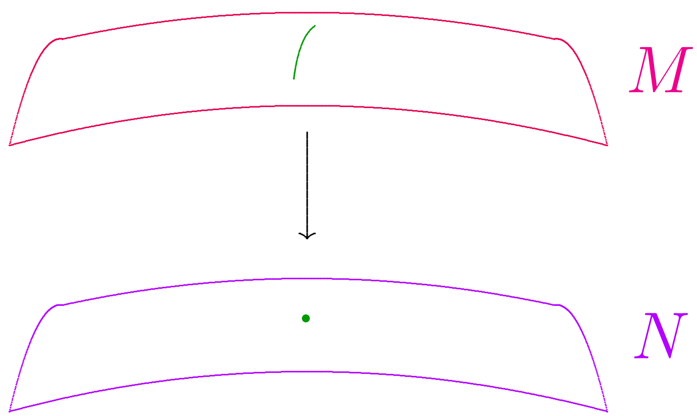


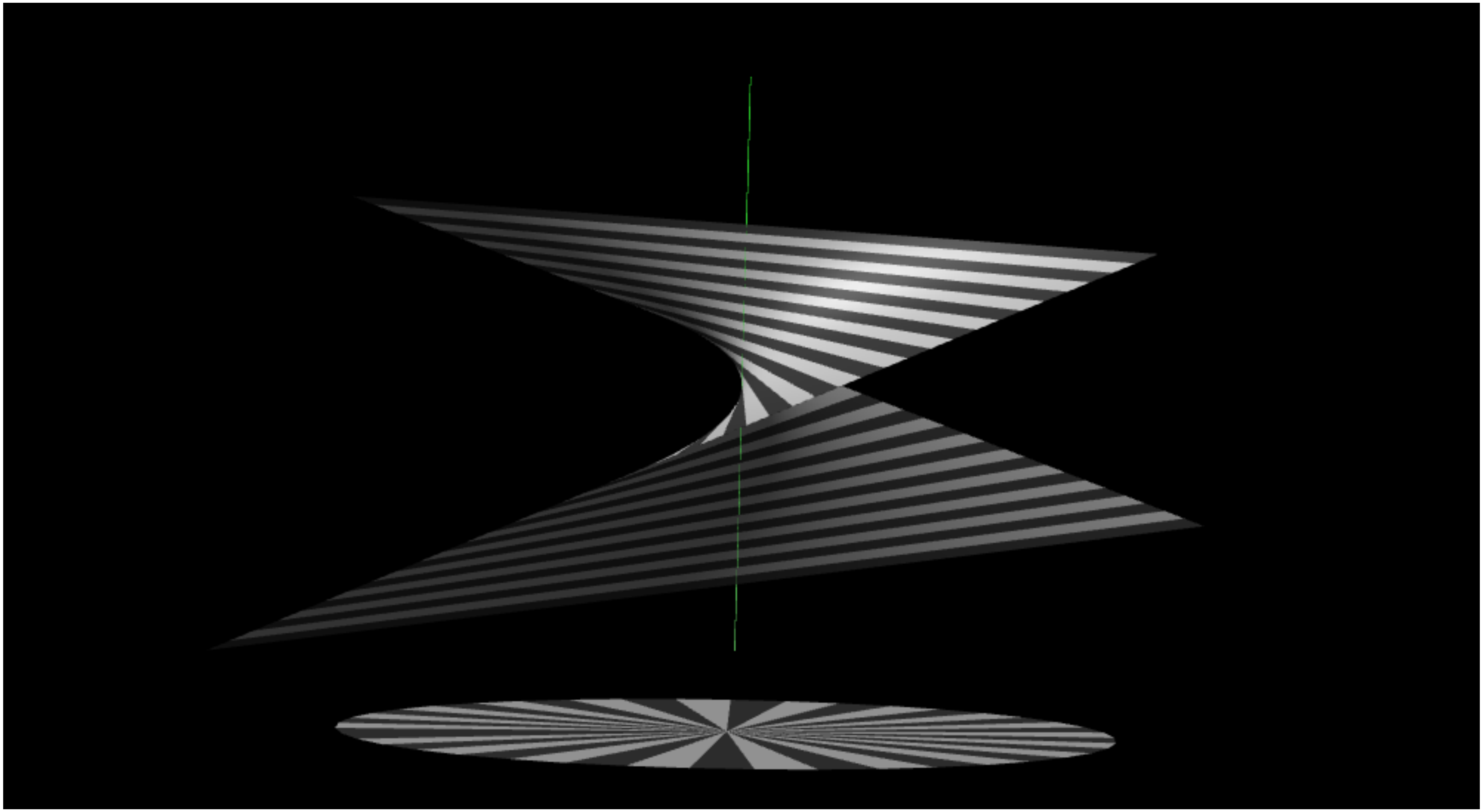
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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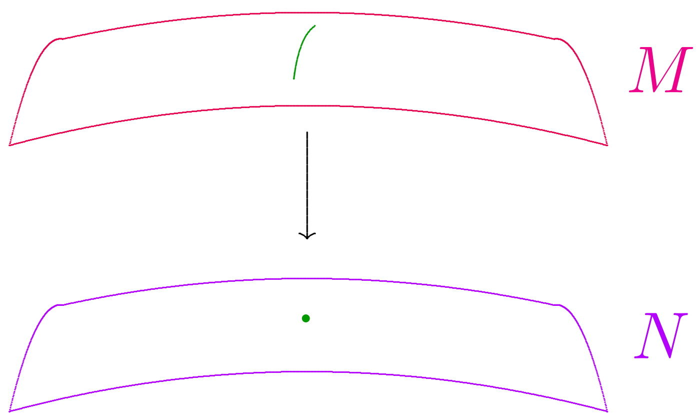

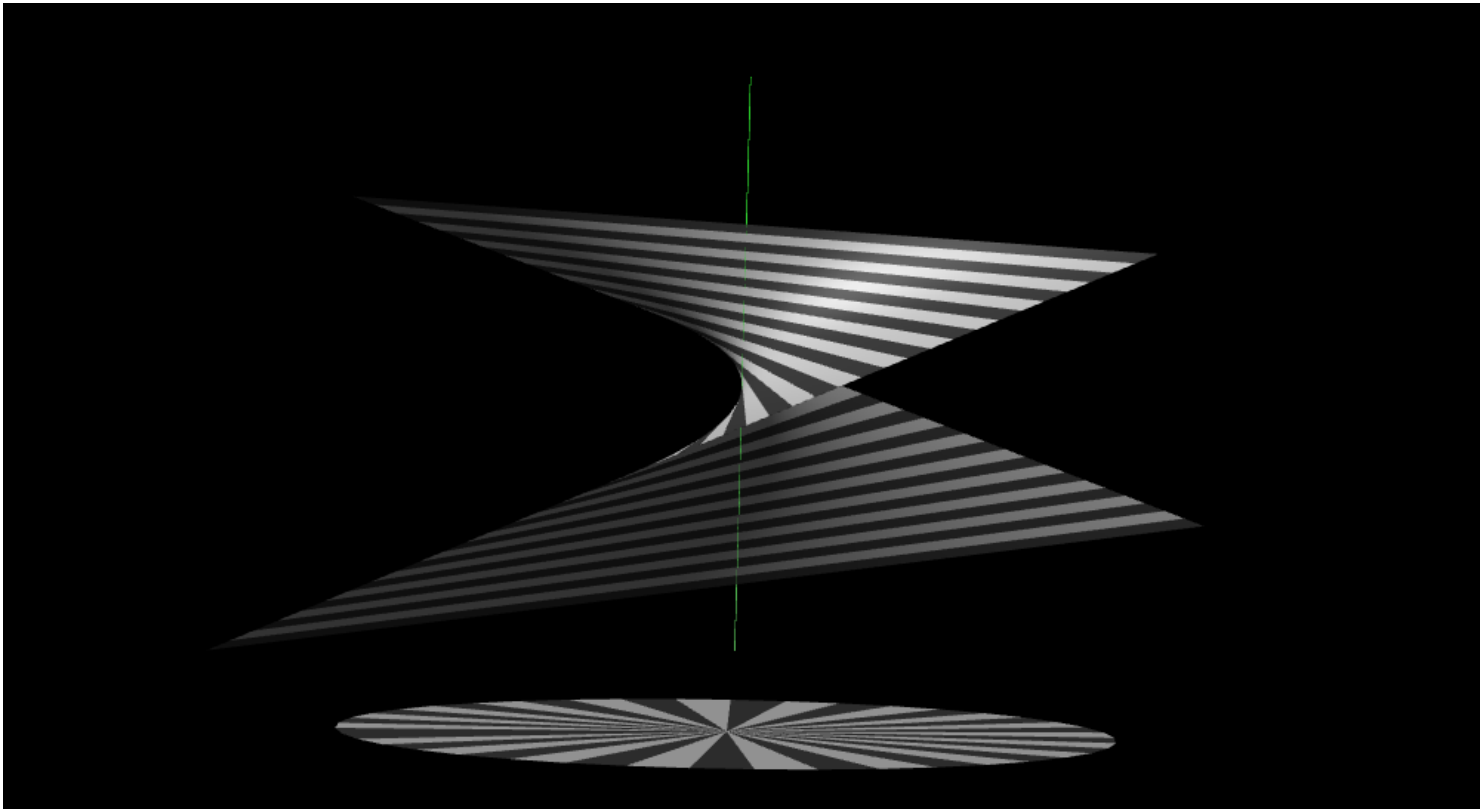

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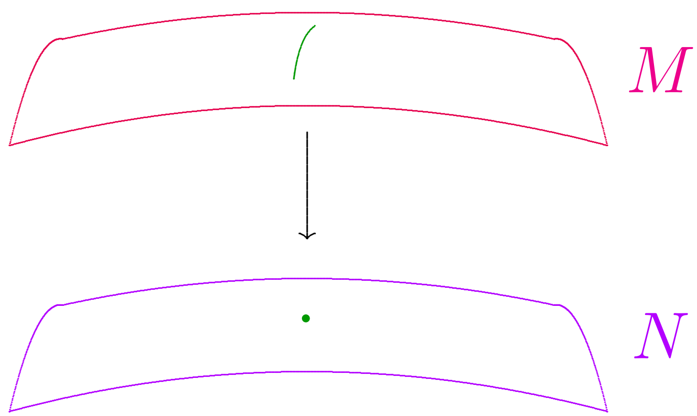

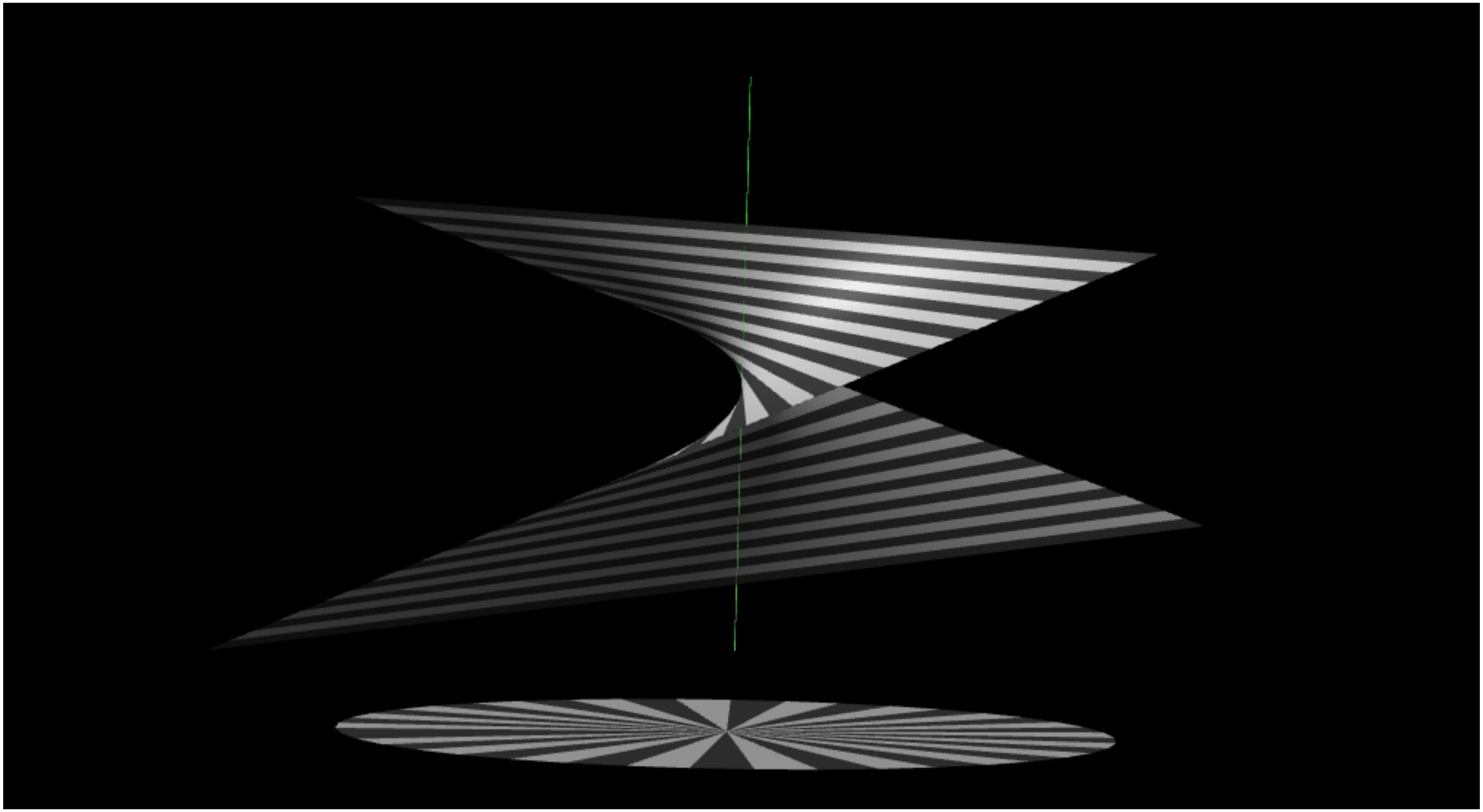

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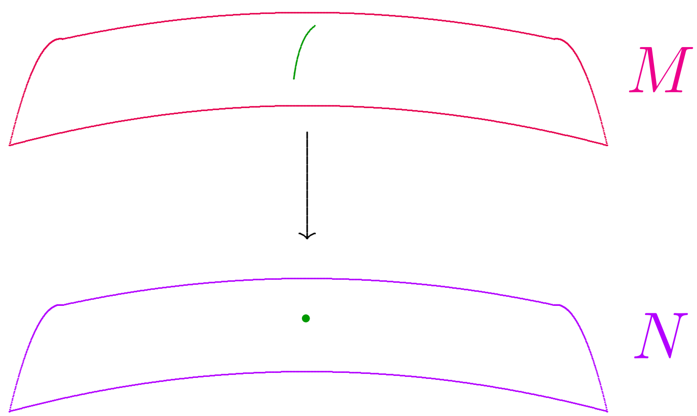

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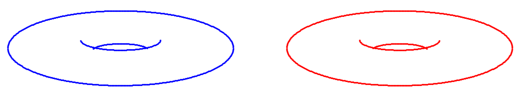

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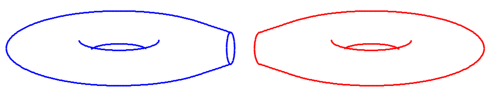

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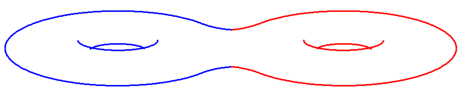

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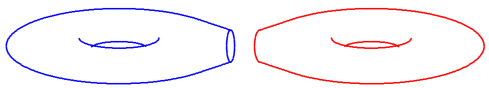

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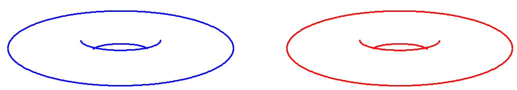

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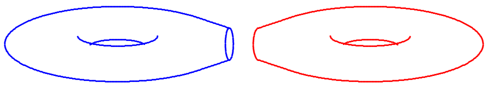

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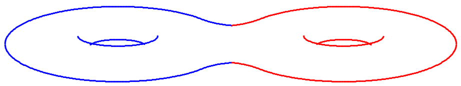

Theorem (CLW '08). *Suppose that M is a smooth compact oriented* 4*-manifold which admits a complex structure J. Then M also admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

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Proof: Seiberg-Witten & Hitchin-Thorpe ineq.

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Proof also uses results of Taubes, McDuff, et al.

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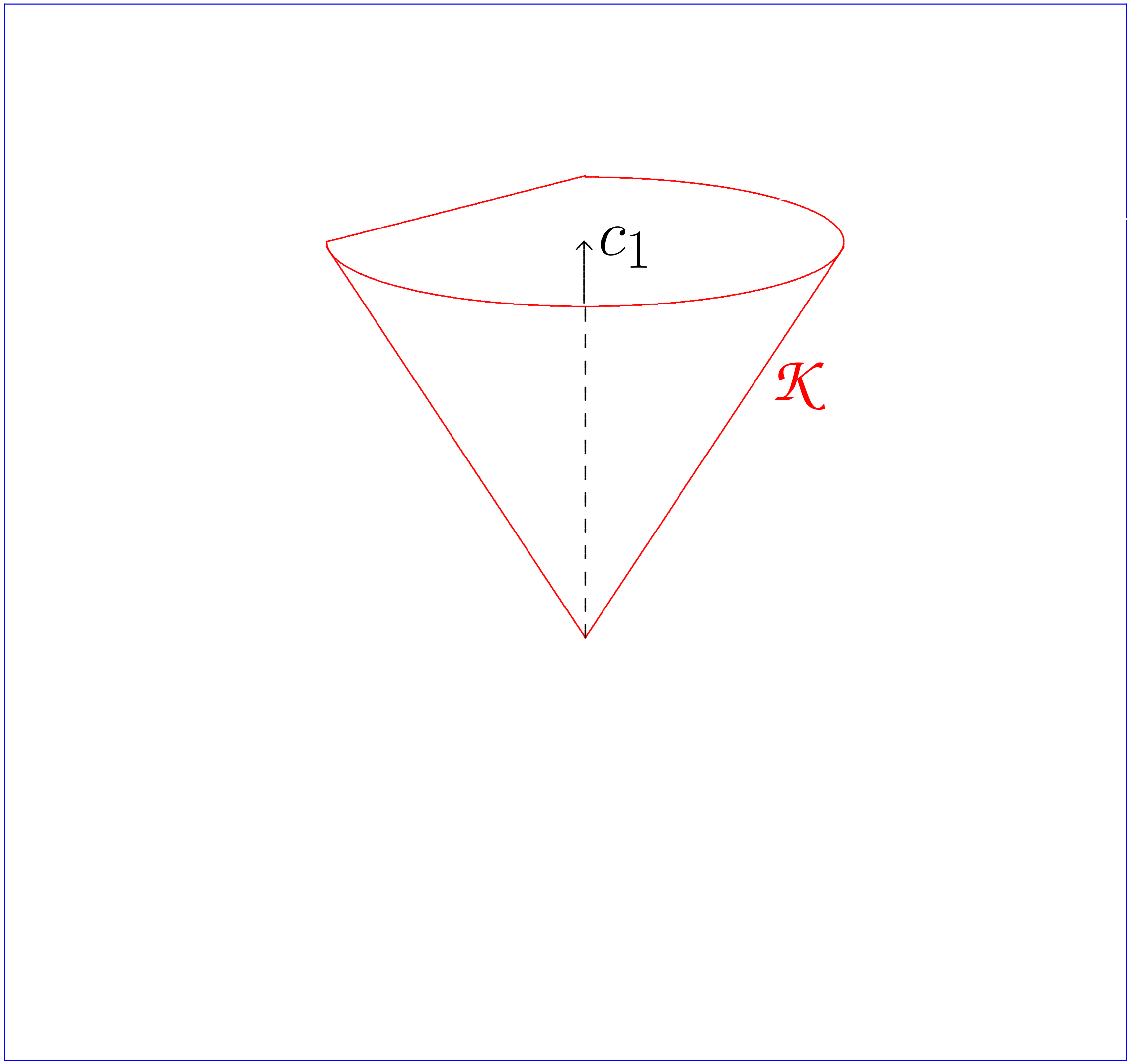
Hard case: when M is toric.

Must understand critical points of

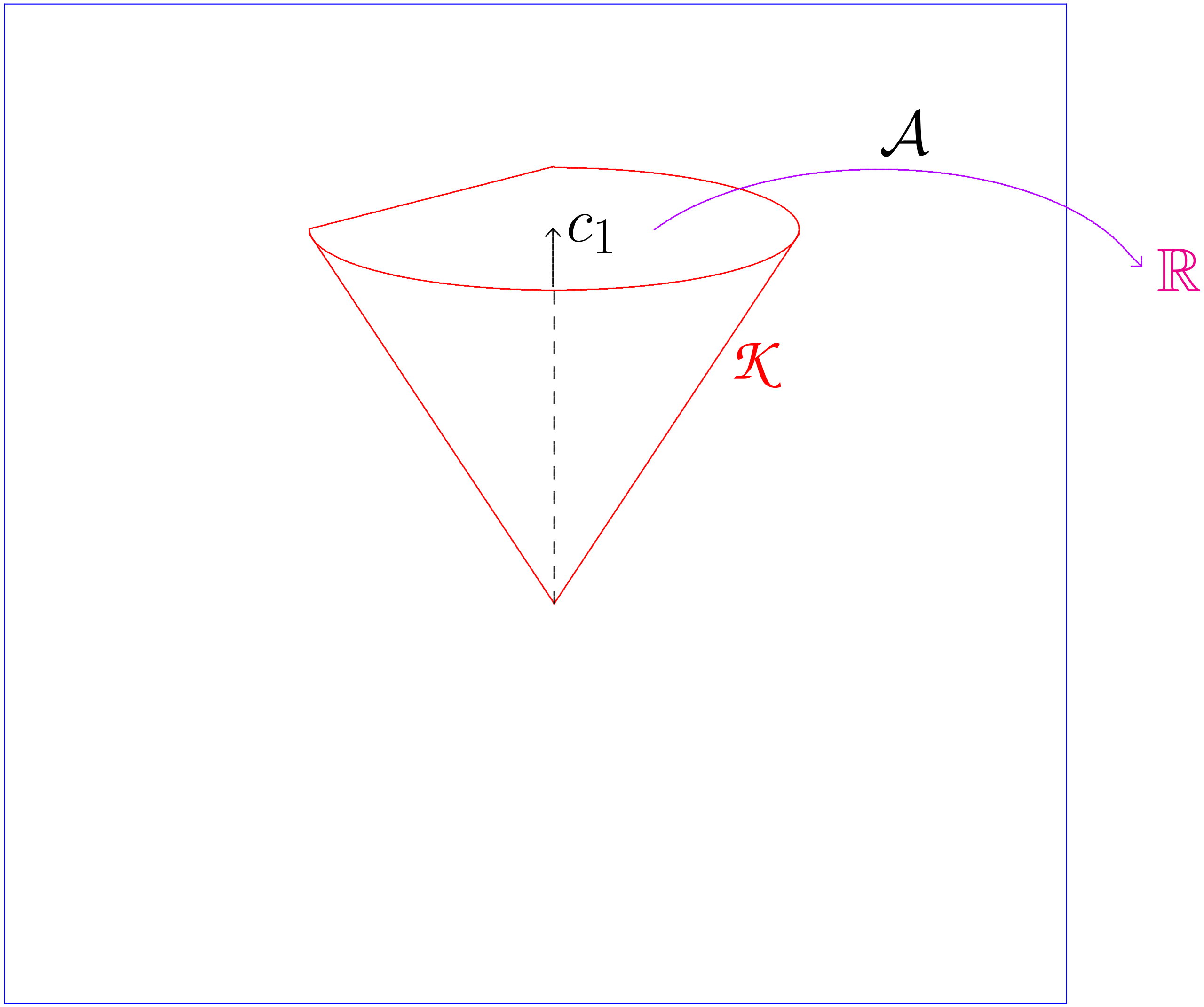
$$\mathcal{A}([\omega]) = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2} \|\mathcal{F}_{[\omega]}\|^2$$

where $\mathcal{F}$ is Futaki invariant.



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$$

$(M$ Del Pezzo$)$



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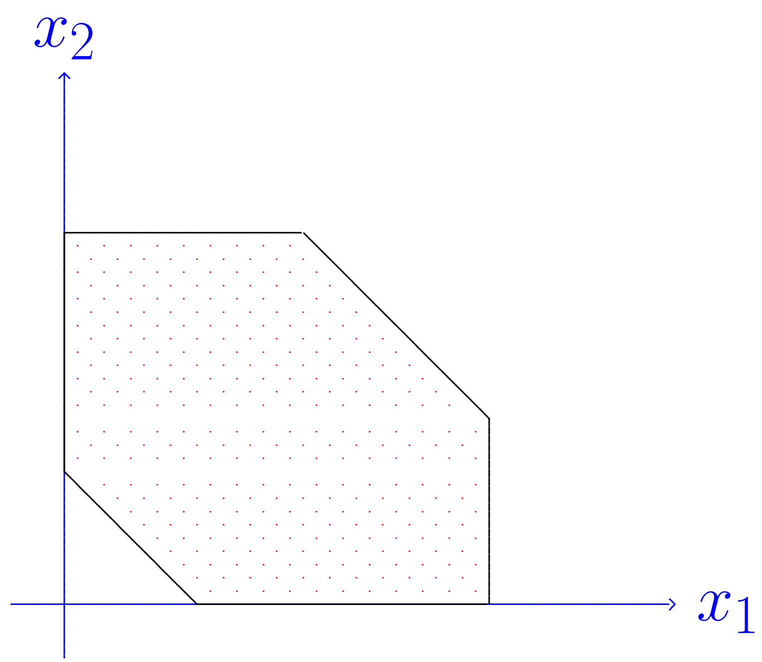
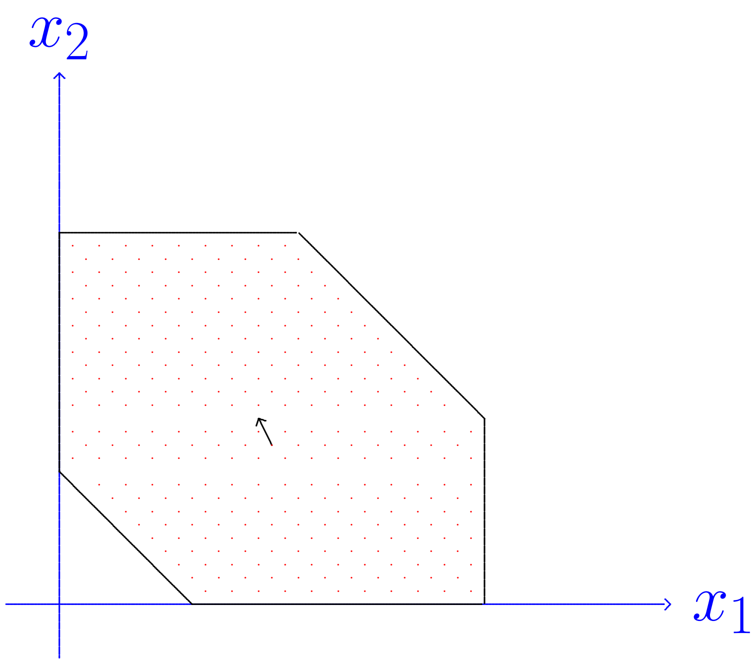
The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon.



The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon. Formula involves barycenters, moments of inertia.



$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

$\mathcal{A}$ is explicit rational function —

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but quite complicated!

$$3\Big[3+28\gamma+96\gamma^2+168\gamma^3+164\gamma^4+80\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^4+16\alpha^6(1+\beta+\gamma)^4+16\beta^5(5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5)+4\beta^4(41+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+$$

$$60\gamma^5+4\gamma^6)+8\beta^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6)+4\beta(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6)+4\beta^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+$$

$$172\gamma^5+24\gamma^6)+16\alpha^5(5+2\beta^5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5+\beta^4(15+14\gamma)+\beta^3(37+70\gamma+30\gamma^2)+\beta^2(43+123\gamma+108\gamma^2+30\gamma^3)+\beta(24+92\gamma+123\gamma^2+70\gamma^3+$$

$$14\gamma^4))+4\alpha^4(41+4\beta^6+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+60\gamma^5+4\gamma^6+\beta^5(60+56\gamma)+\beta^4(263+476\gamma+196\gamma^2)+8\beta^3(62+169\gamma+139\gamma^2+35\gamma^3)+2\beta^2(239+876\gamma+1089\gamma^2+$$

$$556\gamma^3+98\gamma^4)+4\beta(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5))+8\alpha^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6+8\beta^6(1+\gamma)+2\beta^5(37+70\gamma+30\gamma^2)+4\beta^4(62+$$

$$169\gamma+139\gamma^2+35\gamma^3)+4\beta^3(98+353\gamma+428\gamma^2+210\gamma^3+35\gamma^4)+2\beta^2(163+735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+\beta(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+$$

$$8\gamma^6))+4\alpha(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^3+4\beta^5(24+92\gamma+123\gamma^2+70\gamma^3+14\gamma^4)+4\beta^4(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5)+$$

$$2\beta^3(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+8\gamma^6)+4\beta^2(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+2\beta(29+210\gamma+556\gamma^2+736\gamma^3+526\gamma^4+184\gamma^5+$$

$$24\gamma^6))+4\alpha^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+172\gamma^5+24\gamma^6+24\beta^6(1+\gamma)^2+4\beta^5(43+123\gamma+108\gamma^2+30\gamma^3)+2\beta^4(239+876\gamma+1089\gamma^2+556\gamma^3+98\gamma^4)+4\beta^3(163+$$

$$735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+4\beta(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+\beta^2(479+2580\gamma+5058\gamma^2+4716\gamma^3+2178\gamma^4+432\gamma^5+24\gamma^6))\Big]\Big/$$

$$\Big[1+10\gamma+36\gamma^2+64\gamma^3+60\gamma^4+24\gamma^5+24\beta^5(1+\gamma)^5+24\alpha^5(1+\beta+\gamma)^5+12\beta^4(1+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3)+16\beta^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5)+$$

$$12\beta^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5)+2\beta(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5)+12\alpha^4(1+\beta+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3+10\beta^3(1+\gamma)+\beta^2(23+46\gamma+$$

$$16\gamma^2)+2\beta(10+30\gamma+23\gamma^2+5\gamma^3))+16\alpha^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5+15\beta^5(1+\gamma)^2+3\beta^4(19+57\gamma+50\gamma^2+13\gamma^3)+3\beta^3(30+120\gamma+155\gamma^2+78\gamma^3+$$

$$13\gamma^4)+3\beta^2(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+\beta(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5))+12\alpha^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5+20\beta^5(1+\gamma)^3+$$

$$\beta^4(68+272\gamma+366\gamma^2+200\gamma^3+36\gamma^4)+4\beta^3(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+2\beta(12+84\gamma+207\gamma^2+240\gamma^3+136\gamma^4+30\gamma^5)+\beta^2(69+414\gamma+864\gamma^2+$$

$$824\gamma^3+366\gamma^4+60\gamma^5))+2\alpha(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5+60\beta^5(1+\gamma)^4+12\beta^4(15+75\gamma+136\gamma^2+114\gamma^3+43\gamma^4+5\gamma^5)+12\beta^2(12+84\gamma+207\gamma^2+$$

$$240\gamma^3+136\gamma^4+30\gamma^5)+8\beta^3(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5)+3\beta(15+120\gamma+336\gamma^2+448\gamma^3+300\gamma^4+80\gamma^5))\Big]$$

Theorem (L '12). *Let g be Einstein, Hermitian metric on be a Del Pezzo surface (M^4, J). Then $[g]$ minimizes $\int_M |W_+|^2 d\mu$ among all conformally Kähler metrics on (M^4, J).*

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For proof, see `arXiv:1310.0848 [math.DG]`

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Key inequality:

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}\mathcal{A}([\omega]),$$

with equality only if $[\tilde{g}]$ contains extremal Kähler metrics.

Conjecture. *If* M^4 *admits an Einstein, Hermitian metric* g *with* $\lambda > 0$, *then* $[g]$ *minimizes* $\int_M |W_+|^2 d\mu$ *among all conformal classes on* M.

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Non-Kähler cases: eliminate toric condition?