Einstein Metrics,

Four-Manifolds, &

Gravitational Instantons

Claude LeBrun
Stony Brook University

Complex Hermitian Geometry,
Colloque en l'honneur de Paul Gauduchon,
Angers, France, 22 mai, 2025.

Will discuss joint results with

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Olivier Biquard
Sorbonne Université

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(2024) 13295-13311.

Will also briefly mention

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Tristan Ozuch
MIT

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— A. Einstein, to G. Gamow

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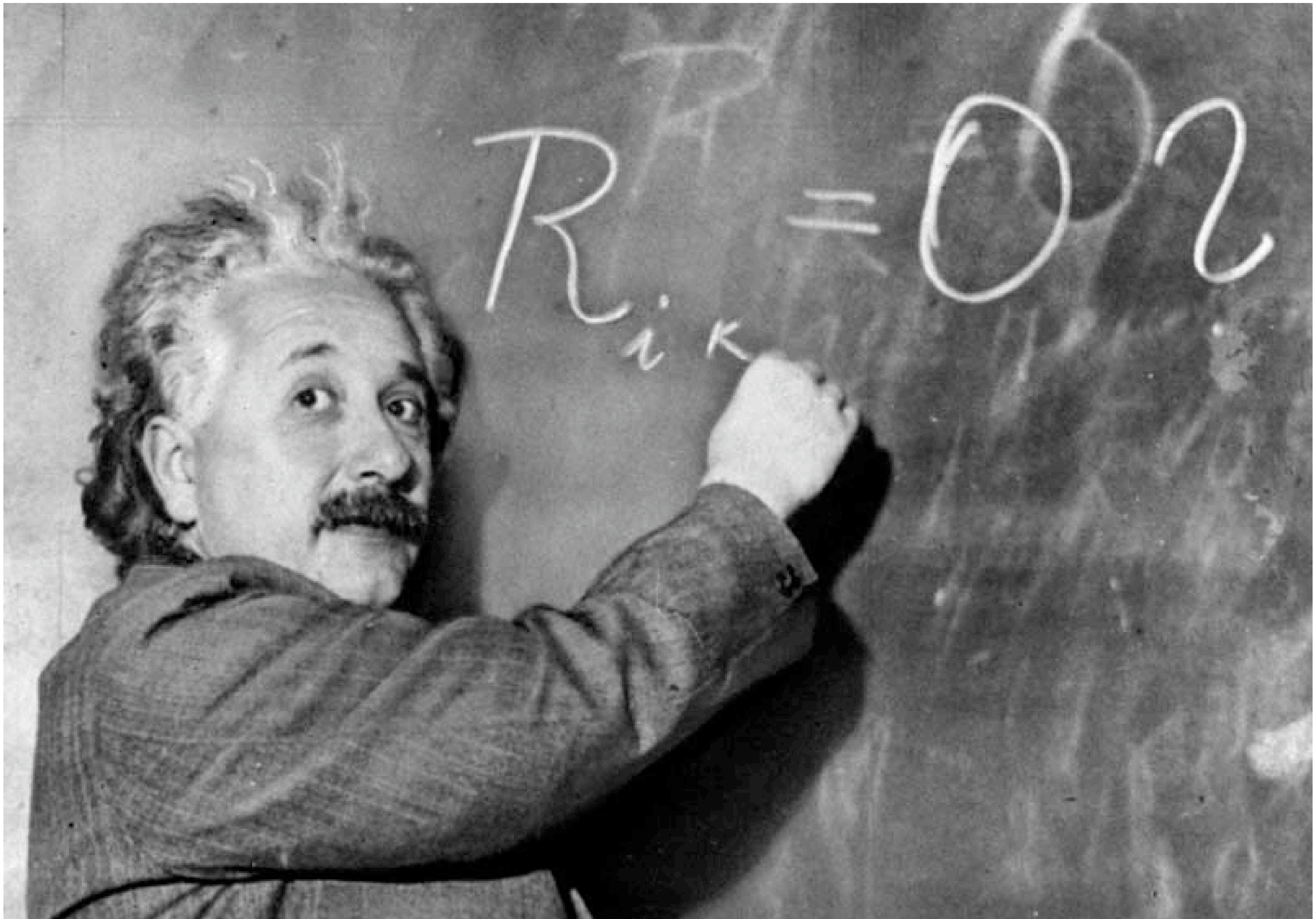

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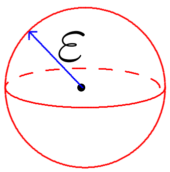

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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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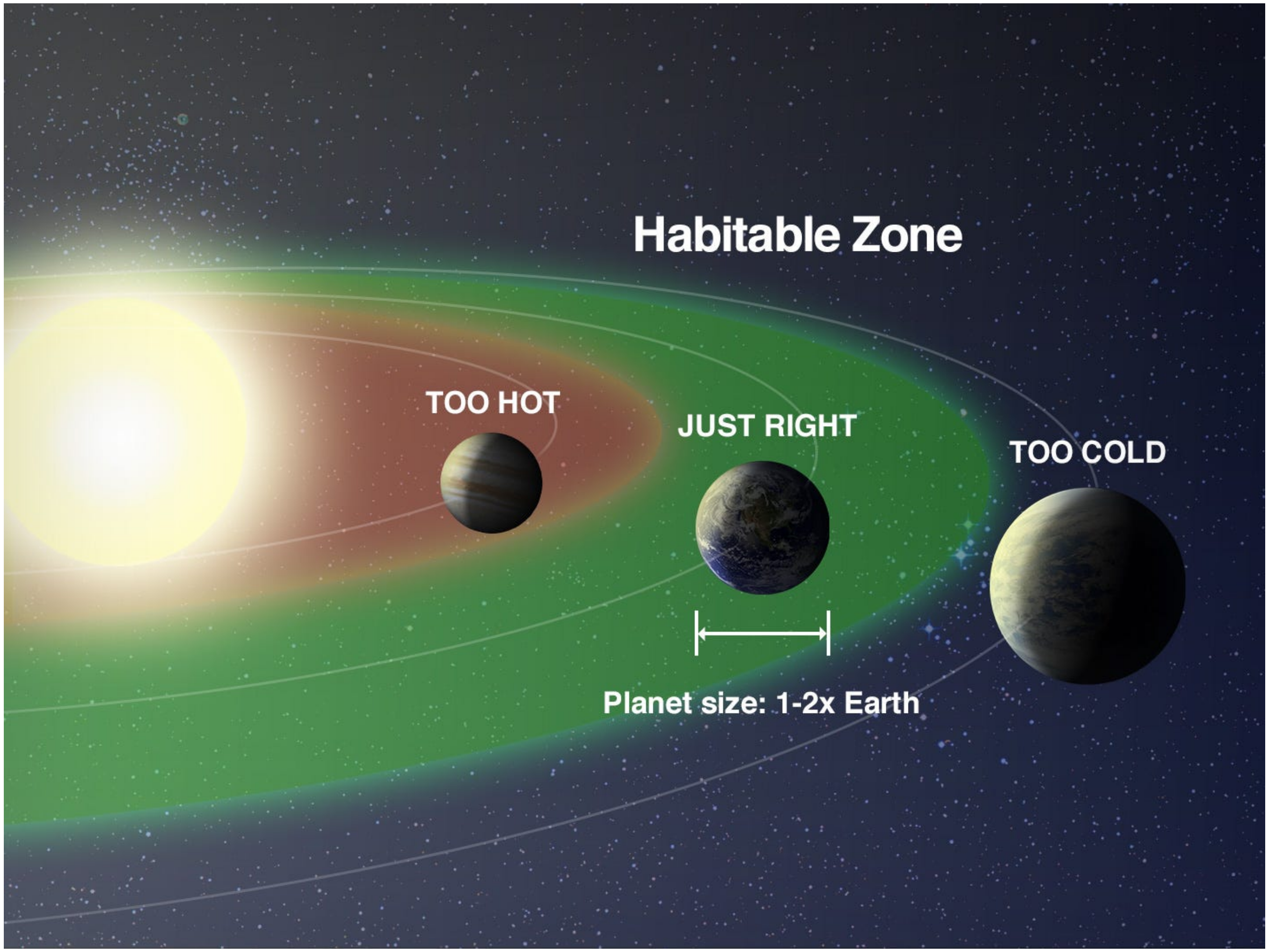

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Λ^+ self-dual 2-forms.
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Which 4-manifolds admit Einstein metrics?

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$$d\omega = 0, \qquad \lrcorner\omega : TM \xrightarrow{\cong} T^*M.$$

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$$\omega = dx \wedge dy + dz \wedge dt$$

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Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* g *(a priori unrelated to* ω*)? What if we also require* $\lambda > 0$*?*

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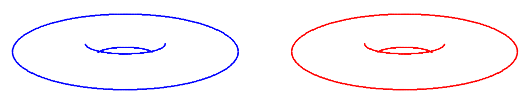

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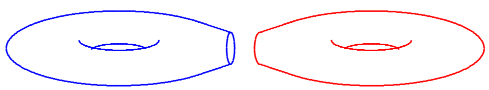

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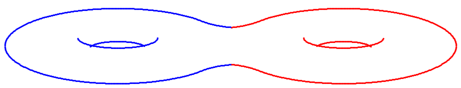

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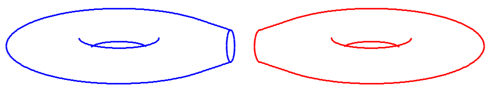

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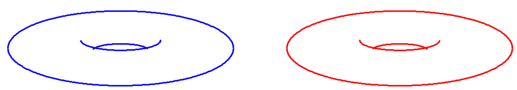

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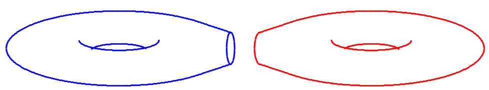

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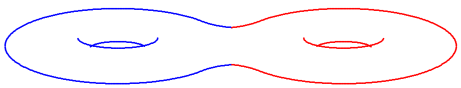

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Diffeotypes: exactly the Del Pezzo surfaces.

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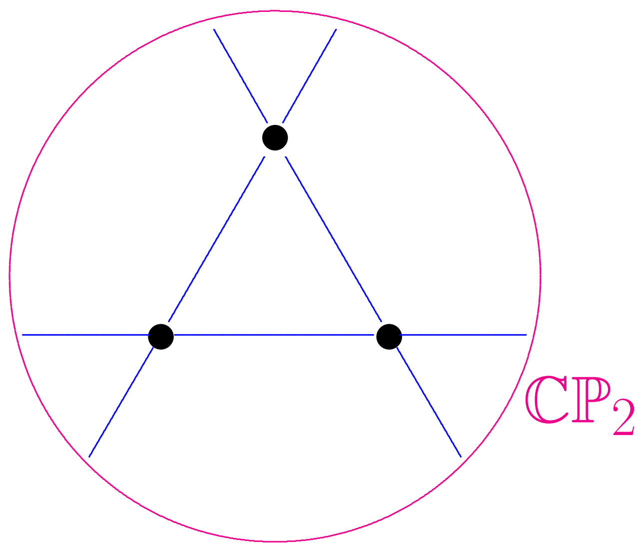

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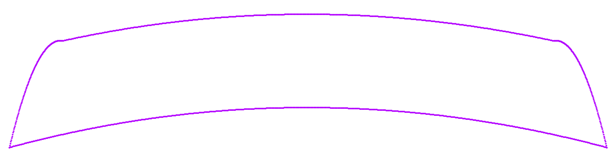

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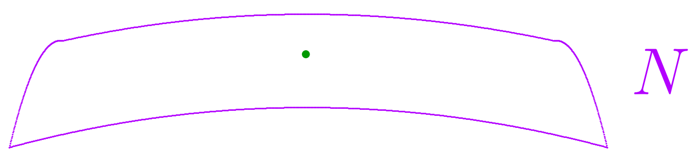

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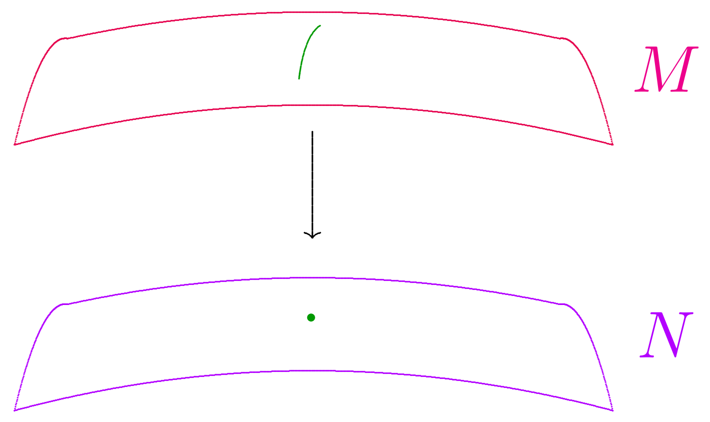

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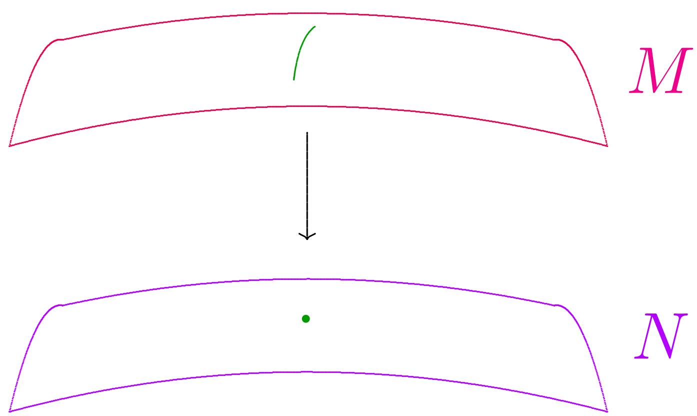

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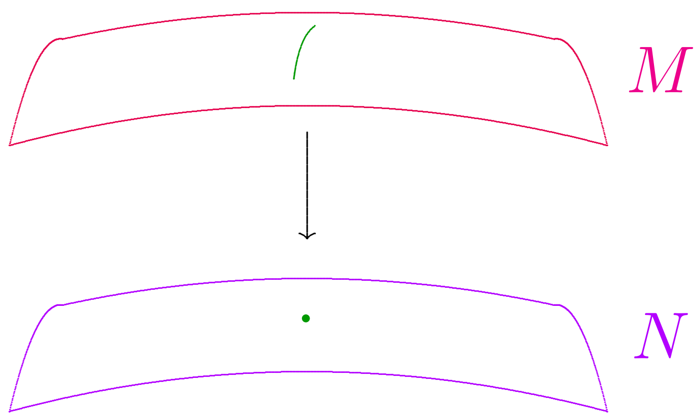

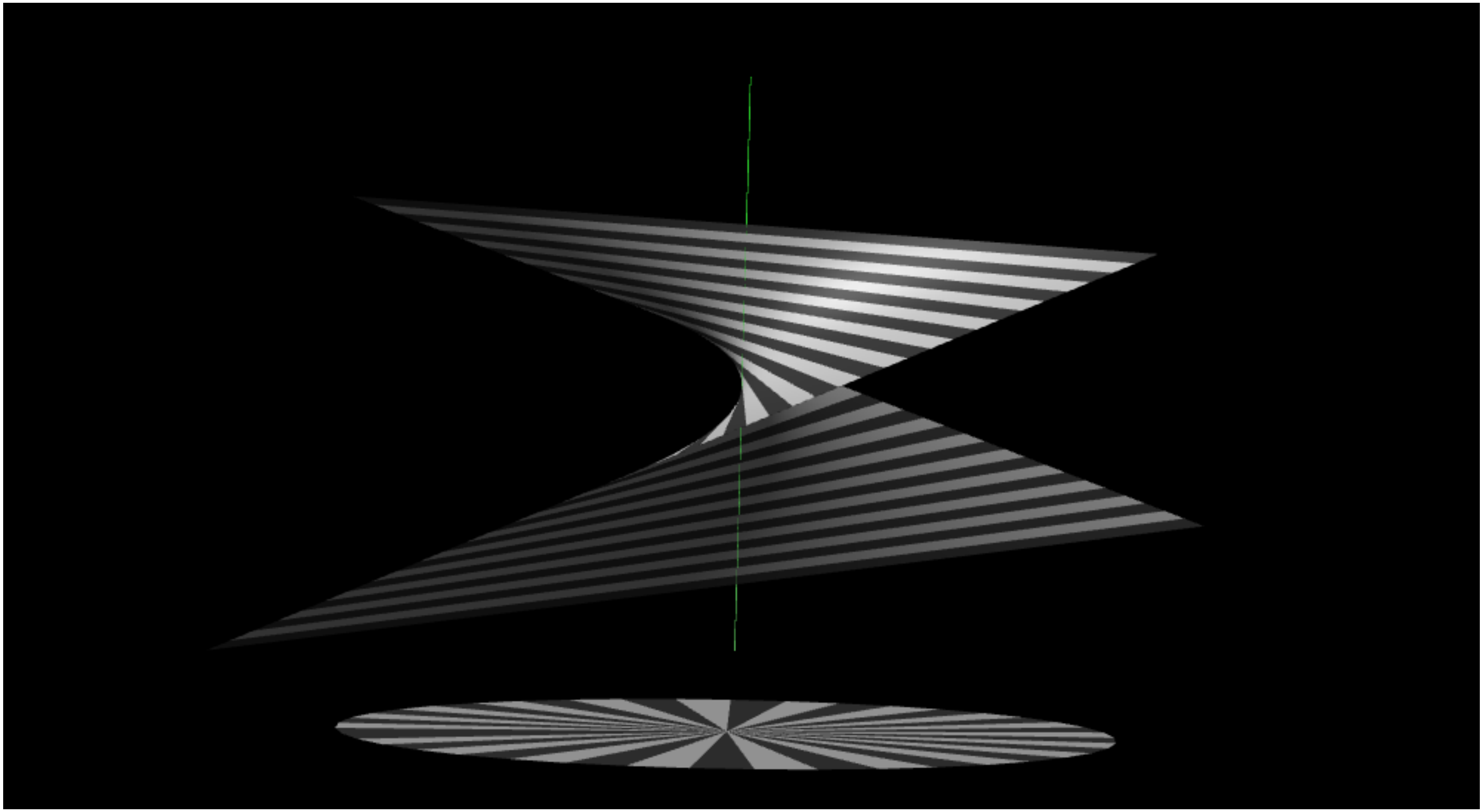

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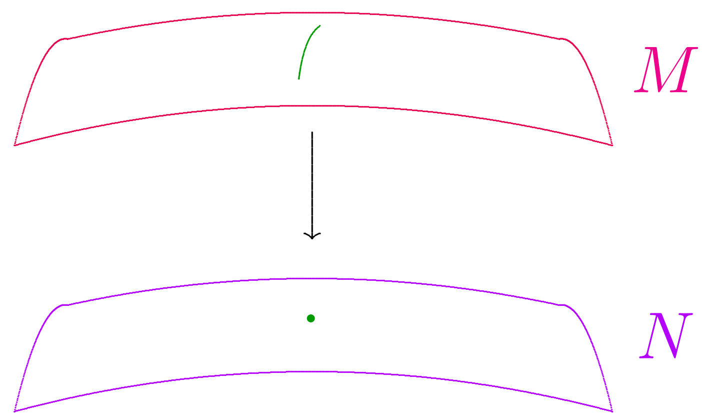

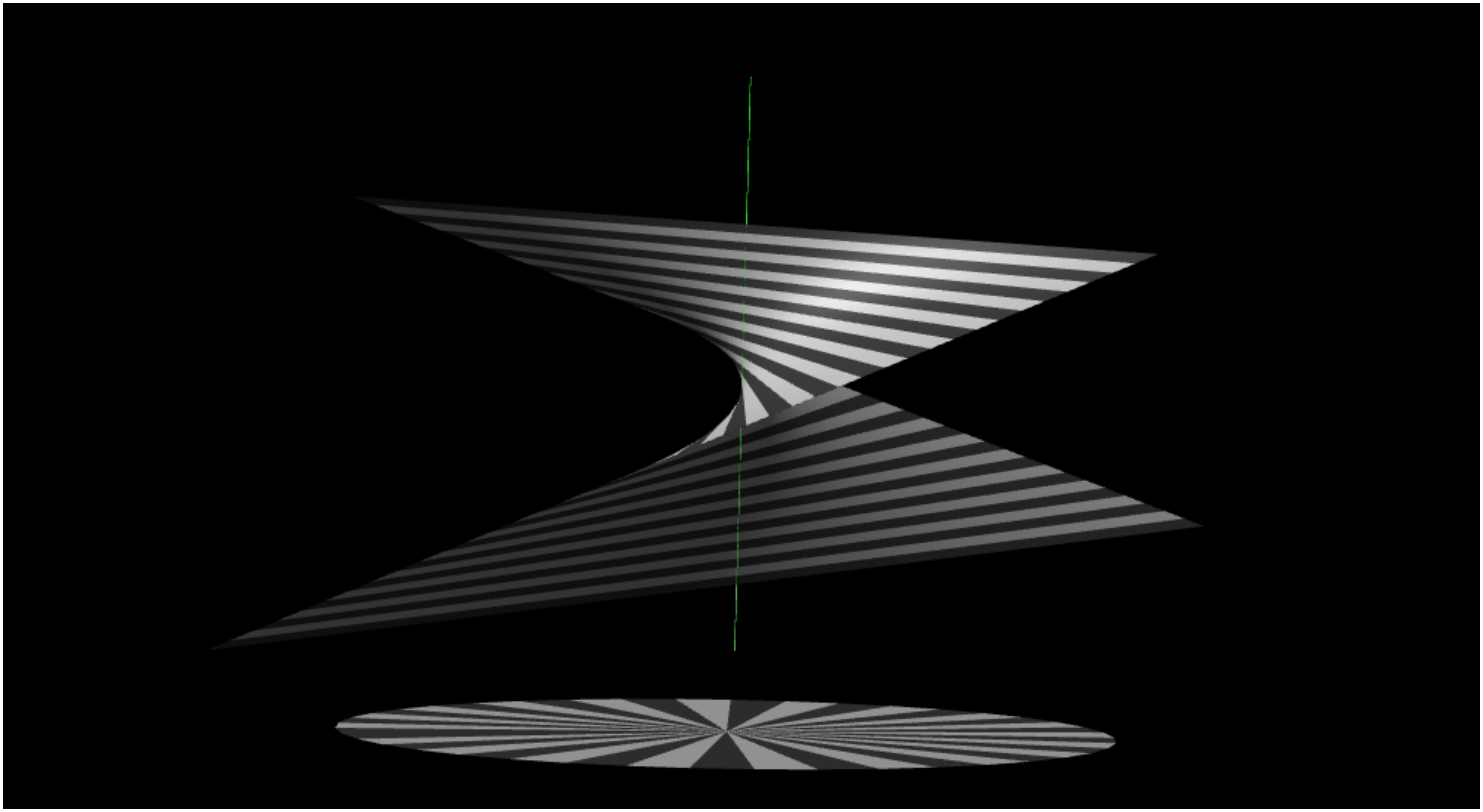

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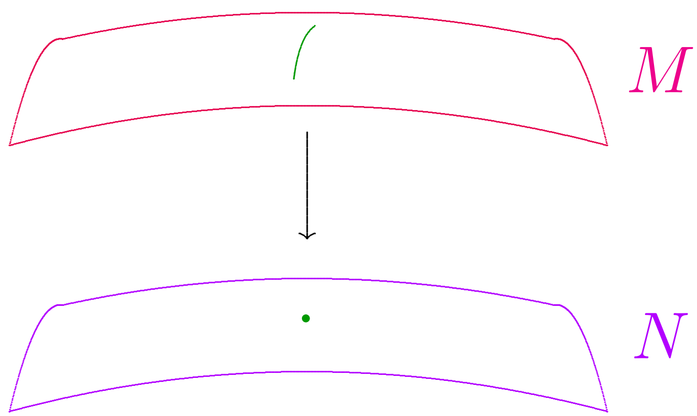
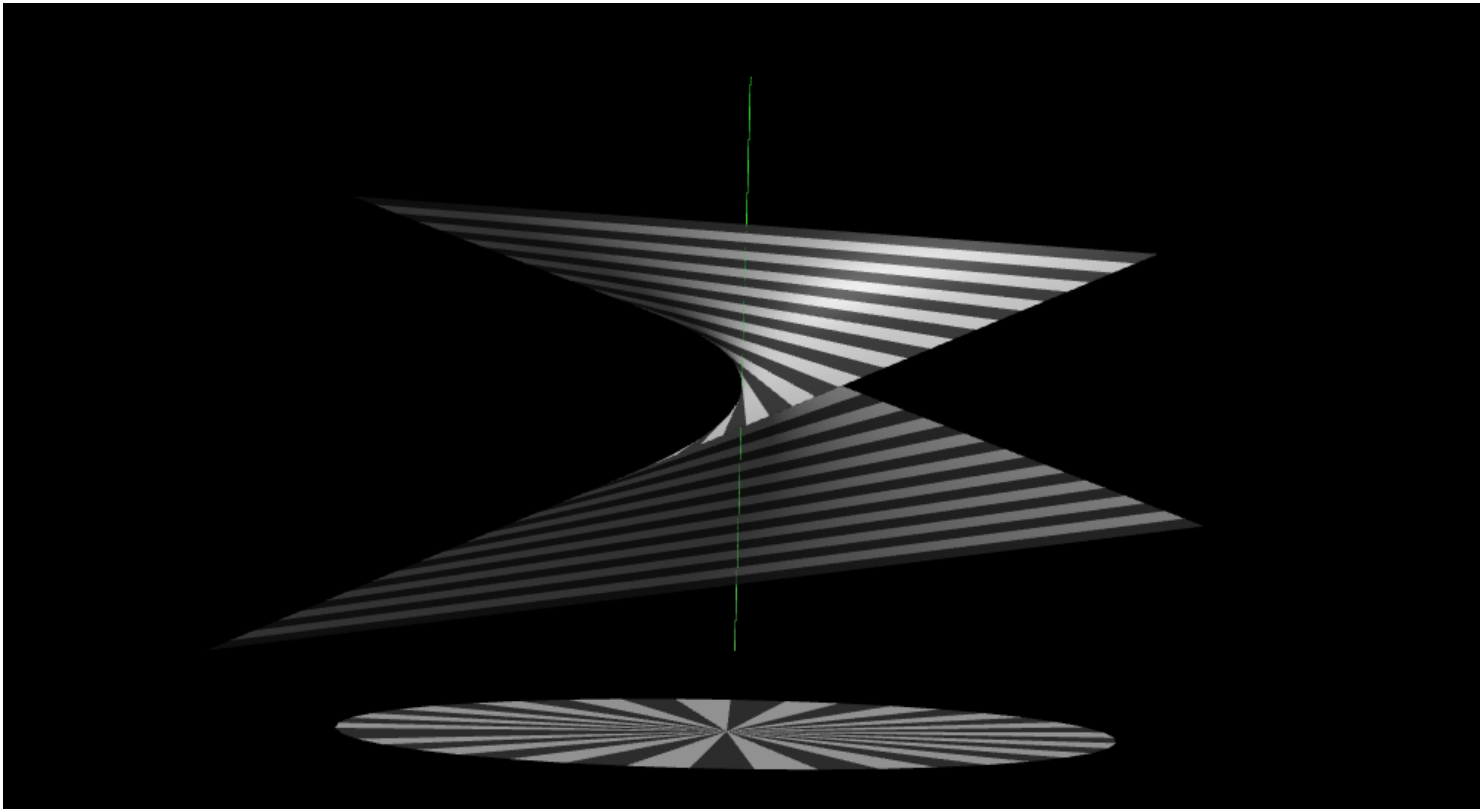


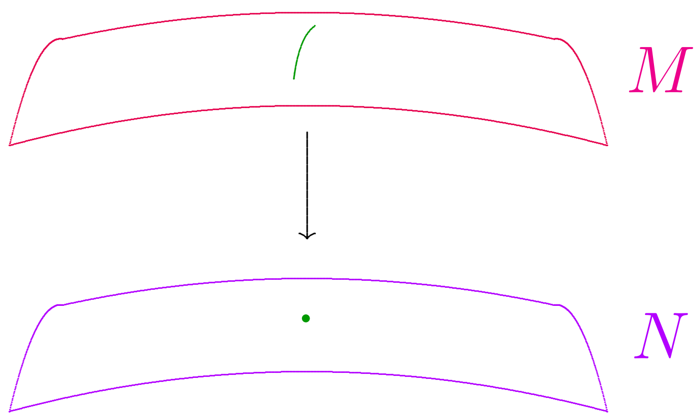


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

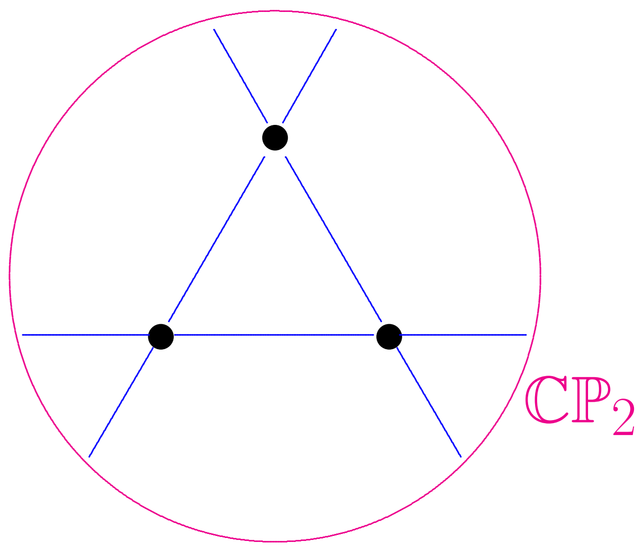
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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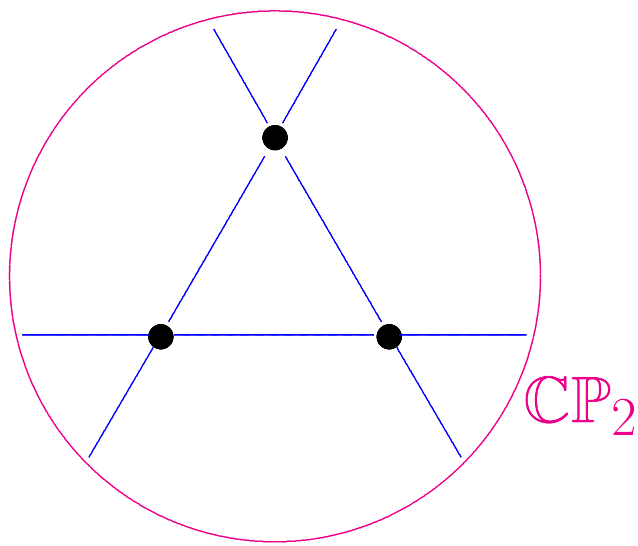

No 3 on a line,

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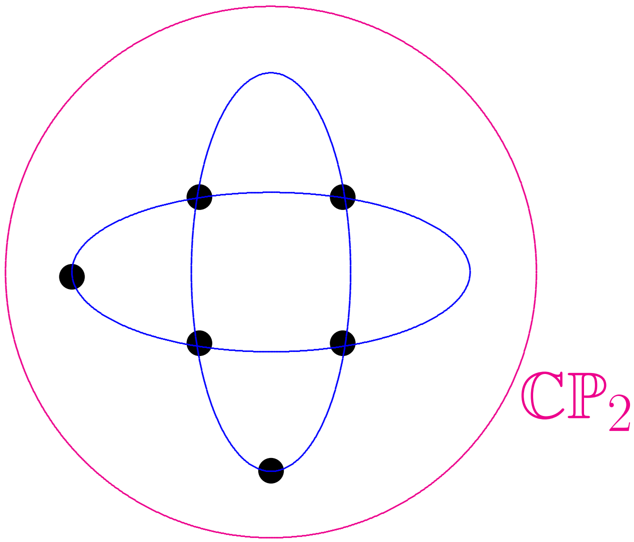

No 3 on a line, no 6 on conic,

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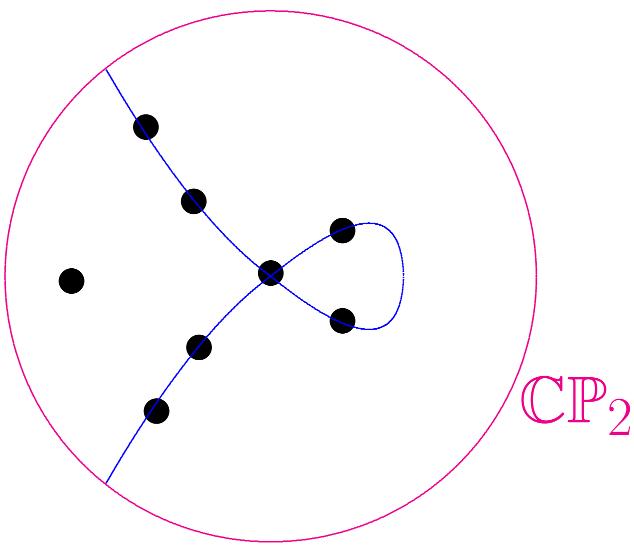

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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$$g = g(J\cdot, J\cdot)$$

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Existence: Page

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

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2008

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Understand all Einstein metrics on del Pezzos.

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L '15 First successful approach, in terms of properties of unique self-dual harmonic 2-form.

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Wu's criterion:

$$\det(W_+) > 0.$$

Theorem (Wu/L '21)**.**

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Top eigenspace $L \subset \Lambda^+$ of W_+ is a line bundle.

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface.*

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W_+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$*.*

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Assume they satisfy: $\quad W_+ \equiv 0.$

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This illustrates how gravitational instantons play a crucial role, even when studying compact case.

Gravitational Instantons?

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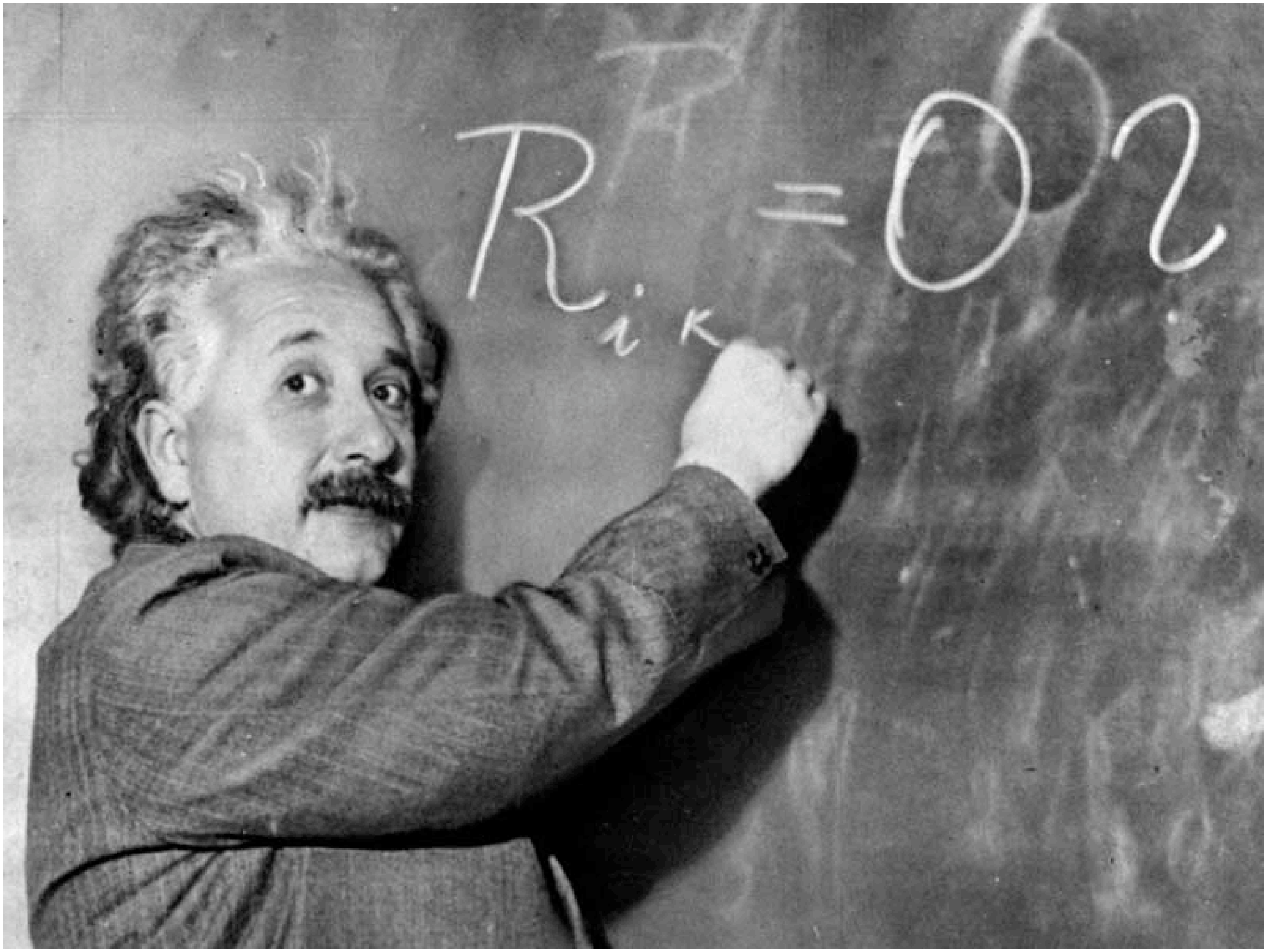

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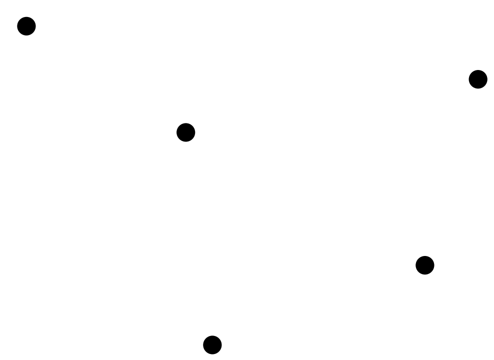
Data: ℓ points in $\mathbb{R}^3$ and a constant $\kappa^2 \geq 0$.



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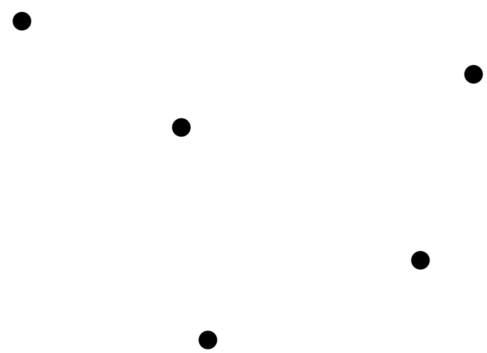

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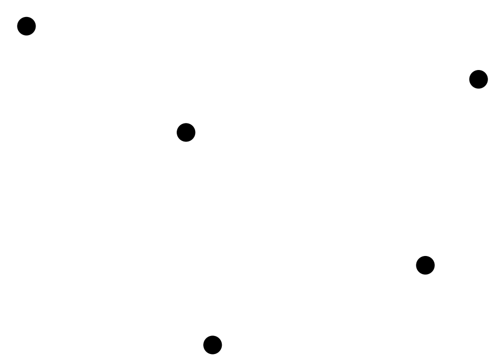

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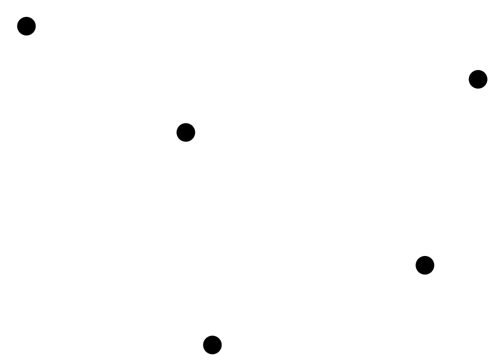

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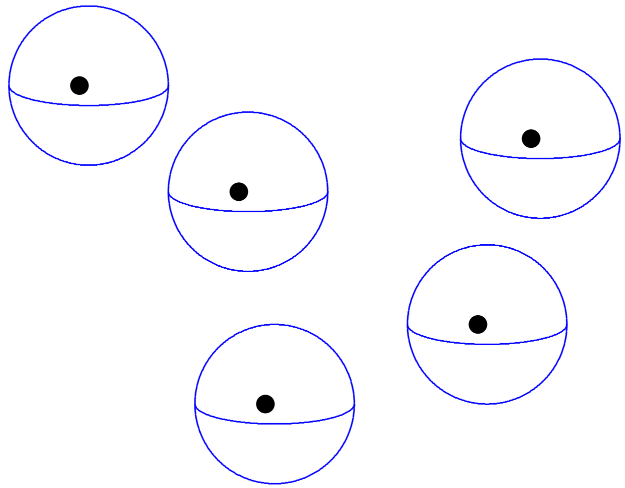

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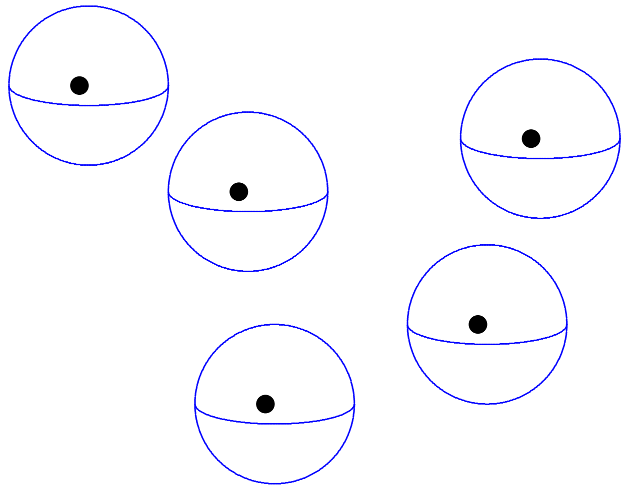

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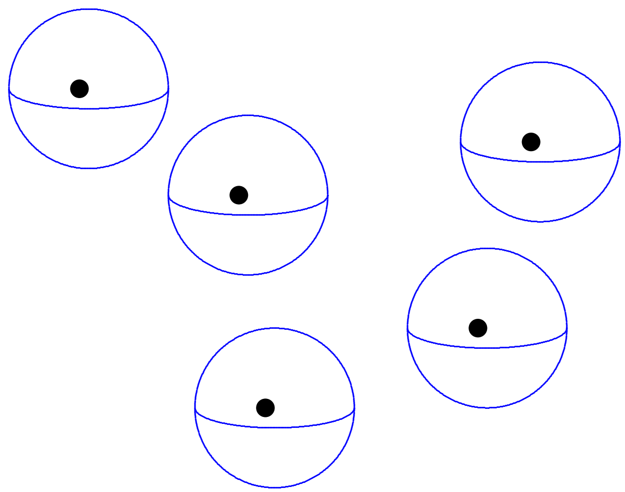

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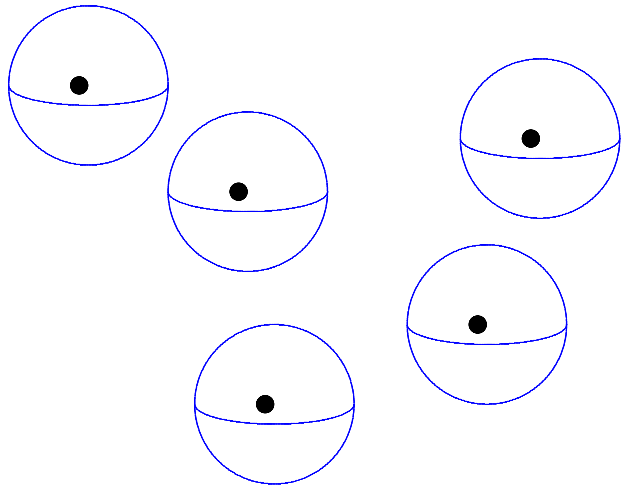

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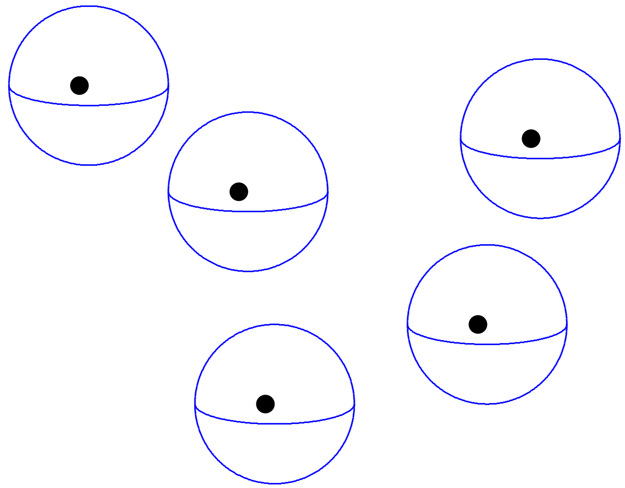

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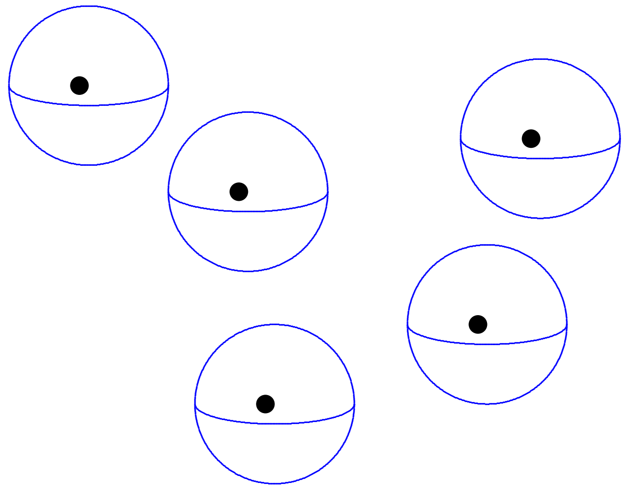

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Gibbons and Hawking were unaware of all this!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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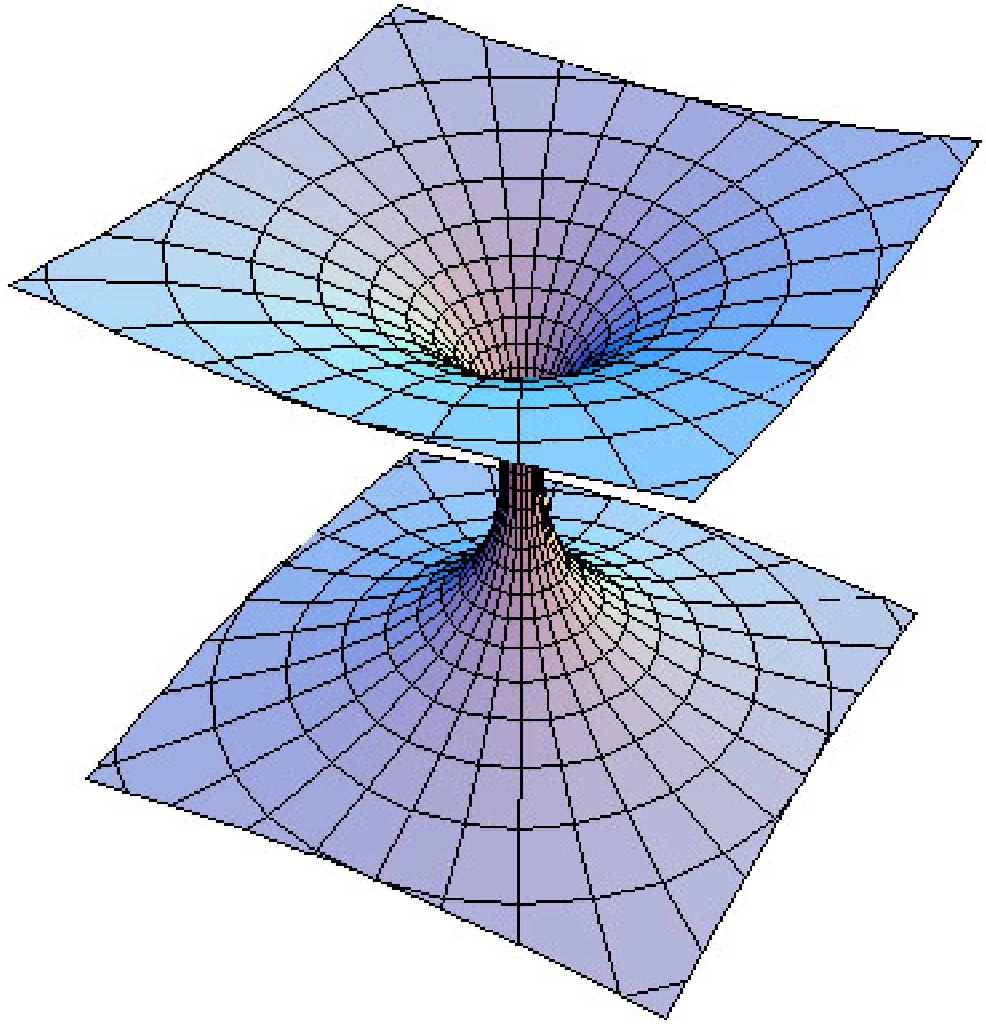
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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This might lend some credence to the aphorism. . .

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"Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes."

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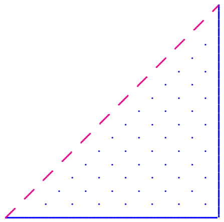

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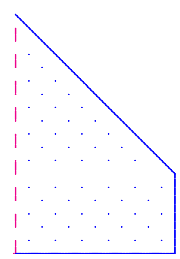

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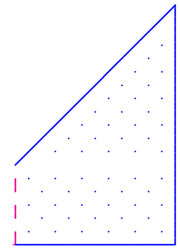

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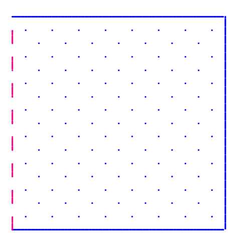

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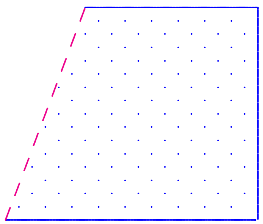

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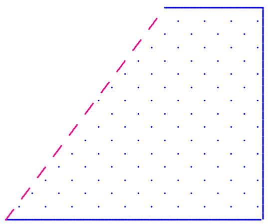

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Y. Chen & E. Teo, 2011

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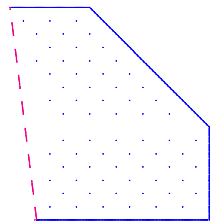

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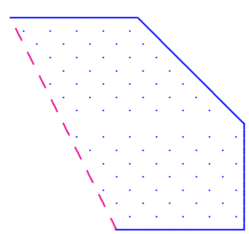

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at every point, with respect to h. Now use Stokes!

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With appropriate fall-off at infinity, this can be used to control the boundary term.

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This optimal result combines **Theorem A** with a result of Mingyang Li, `arXiv:2310.13197`.

Merci de m'avoir invité!

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Je suis ravi d'être ici!

Félicitations, Paul!

Et Bon Anniversaire!