MAT 569 Homework

HW # 2

1. Let G be a compact Lie group, and recall that any such G carries a Riemannian metric g that is *bi-invariant*, in the sense that it is preserved by both left and right translations of G. Also recall that the Lie algebra $\mathfrak{g}$ of G consists of all vector field on G that are invariant under *left* translations.

(a) Prove that the Riemannian connection of g satisfies

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

for all $X, Y \in \mathfrak{g}$.

(b) Use this to show that the Riemann curvature tensor $\mathcal{R}$ of g satisfies

$$\mathcal{R}(\_, Z, X, Y) = \frac{1}{4}[Z, [X, Y]]$$

for all $X, Y, Z \in \mathfrak{g}$.

(c) If $X, Y \in \mathfrak{g}$ are normalized so that $g(X, X) = g(Y, Y) = 1$ and $g(X, Y) = 0$, use (b) to show that the sectional curvature of (G, g) in the plane spanned by X and Y is given by

$$K(X, Y) = \frac{1}{4}\|[X, Y]\|^2 = \frac{1}{4}g([X, Y], [X, Y]).$$

Conclude that the sectional curvature of (G, g) is necessarily ≥ 0.

(d) For every $X \in \mathfrak{g}$, there is an associated linear transformation $\mathrm{Ad}_X : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\mathrm{Ad}_X(Y) = [X, Y].$$

Use (b) to show that the Ricci tensor of g is given by $r = -\frac{1}{4}B$, where the *Killing form* $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is defined to be

$$B(X, Y) = \mathrm{Tr}(\mathrm{Ad}_X \circ \mathrm{Ad}_Y),$$

where Tr denotes the trace of a linear transformation $\mathfrak{g} \to \mathfrak{g}$.

2. Let (M, g) be a smooth Riemannian n-manifold with non-negative Ricci curvature, and suppose that there is a compact subset $K \subset M$ such that $M - K$ is isometric to the complement of a closed ball in Euclidean n-space. By replacing a ball in some flat n-torus T^n with a neighborhood U of K, construct a compact Riemannian n-manifold (N, h) with non-negative Ricci curvature and $b_1 \geq n$. Using Bochner's theorem, prove that (N, h) is flat, and then use this to prove that (M, g) is isometric to Euclidean n-space.

3. Let (M, g) be as in problem 2. Use the splitting theorem to give a different proof of the fact that (M, g) is isometric to Euclidean n-space.

4. Let (M, g) once again be as in problem 2, and identify $M - K$ isometrically with the complement of the closed ball of some radius ϱ_0 centered at the origin in $\mathbb{R}^n$. Choose any $p \in K$, and show that there is a real number L such that, for all $\varrho \gg 0$, the distance ball $B_\varrho(p, M)$ in M contains the Euclidean annulus $B_{\varrho-L}(0, \mathbb{R}^n) - B_{\varrho_0}(0, \mathbb{R}^n) \subset \mathbb{R}^n$. Use this to show that

$$\liminf_{\varrho \to \infty} \frac{\mathrm{Vol}\, B_\varrho(p, M)}{\mathrm{Vol}\, B_\varrho(0, \mathbb{R}^n)} \geq 1.$$

Then use the Bishop-Gromov inequality to give yet another proof of the fact that (M, g) is isometric to Euclidean n-space.