MAT 569 Homework

HW #2

1. Let (M, g) be a smooth Riemannian n-manifold with non-negative Ricci curvature, and suppose that there is a compact subset $K \subset M$ such that $M - K$ is isometric to the complement of a closed ball in Euclidean n-space. By replacing a ball in some flat n-torus T^n with a neighborhood U of K, construct a compact Riemannian n-manifold (N, h) with non-negative Ricci curvature and $b_1 \geq n$. Using Bochner's theorem, prove that (N, h) is flat, and then use this to prove that (M, g) is isometric to Euclidean n-space.

2. Let (M, g) be as in problem 1. Use the splitting theorem to give a different proof of the fact that (M, g) is isometric to Euclidean n-space.

3. Let (M, g) once again be as in problem 1, and identify $M - K$ isometrically with the complement of the closed ball of some radius ϱ_0 centered at the origin in $\mathbb{R}^n$. Choose any $p \in K$, and show that there is a real number L such that, for all $\varrho \gg 0$, the distance ball $B_\varrho(p, M)$ in M contains the Euclidean annulus $B_{\varrho - L}(0, \mathbb{R}^n) - B_{\varrho_0}(0, \mathbb{R}^n) \subset \mathbb{R}^n$. Use this to show that

$$\liminf_{\varrho \to \infty} \frac{\operatorname{Vol} B_\varrho(p, M)}{\operatorname{Vol} B_\varrho(0, \mathbb{R}^n)} \geq 1.$$

Then use the Bishop-Gromov inequality to give yet another proof of the fact that (M, g) is isometric to Euclidean n-space.

4. Let (M, g) be a Riemannian n-manifold of non-negative Ricci curvature, and suppose that there is a diffeomorphism $\Phi : (M - K) \to (\mathbb{R}^n - D)$, for some compact $K \subset M$ and some closed ball $D \subset \mathbb{R}^n$, such that

$$|[(\Phi^{-1})^* g]_{jk} - \delta_{jk}| < A \varrho^{-\tau}$$

for all j, k, where ϱ is the Euclidean radius, and where $A > 0$ and $\tau > 1$ are constants. By modifying your solution to problem 3, show that (M, g) is isometric to Euclidean space.