MAT 569 Homework

HW # 2

1. Let G be a Lie group, and let g be a bi-invariant metric on G. For left-invariant vector fields $X, Y, Z \in \mathfrak{g}$, recall that the connection and curvature of g are then given by

$$\nabla_X Y = \tfrac{1}{2}[X, Y]$$

and

$$\mathcal{R}_{XY} Z = \tfrac{1}{4}[Z, [X, Y]].$$

Use this to show, *by direct calculation*, that $\nabla \mathcal{R} = 0$.

Hint: First observe that

$$(\nabla_W \mathcal{R})_{XY} Z = \nabla_W (\mathcal{R}_{XY} Z) - \mathcal{R}_{(\nabla_W X)Y} Z - \mathcal{R}_{X(\nabla_W Y)} Z - \mathcal{R}_{XY}(\nabla_W Z).$$

Now express this in terms of Lie brackets, and use the Jacobi identity a couple of times.

2. Let G be a compact connected Lie group with bi-invariant metric g. If φ is a harmonic k-form on (G, g), show that φ is bi-invariant. Then, for any $X_0, \ldots, X_k \in \mathfrak{g}$, show that this implies

$$X_0 \, \varphi(X_1, \ldots, X_k) = 0 \qquad (1)$$

$$\sum_{j=1}^{k} \varphi(X_1, \ldots, X_{j-1}, [X_0, X_j], X_{j+1}, \ldots, X_k) = 0 \qquad (2)$$

Hint: For any $a \in G$, argue that $L_a^* \varphi$ and $R_a^* \varphi$ are both cohomologous to φ. Then use this to show that the Lie derivative $\mathcal{L}_{X_0} \varphi$ must vanish.

3. Conversely, suppose that φ is a bi-invariant k-form on a compact Lie group G. Show that φ satisfies equations (1-2), and use this to show that φ is closed.

Now choose an orientation and a bi-invariant metric g on G. Show that $\star\varphi$ is then also bi-invariant. Then use this to show that φ is a harmonic k-form on (G, g).

With problem 2, this shows that *a differential form on a compact Lie group is harmonic iff it is bi-invariant.*

4. Let G be a compact connected Lie group, equipped with bi-invariant metric g. For $X, Y, Z \in \mathfrak{g}$, set

$$\varphi(X, Y, Z) = g(X, [Y, Z]).$$

Show that φ is a bi-invariant 3-form on G. Then use this to show that $H^3(G, \mathbb{R}) \neq 0$ unless G is Abelian.

5. Let $\mathbb{H} \approx \mathbb{R}^4$ denote the quaternions[1]. With quaternionic conjugation defined by $\overline{t\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}} = t\mathbf{1} - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$, show that $q\bar{q} = \|q\|^2\mathbf{1}$. Show that $\overline{q_1 q_2} = \bar{q}_2\bar{q}_1$, and then deduce that quaternionic multiplication satisfies

$$\|q_1 q_2\| = \|q_1\| \, \|q_2\|.$$

Finally, show that this makes the unit sphere $S^3 \subset \mathbb{H}$ into a compact Lie group. This Lie group is usually called $\mathbf{Sp}(1)$.

6. Show that the Lie algebra $\mathfrak{sp}(1)$ of $\mathbf{Sp}(1)$ has a basis X, Y, Z such that

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

Then show that there is a bi-invariant metric for which this basis is orthonormal. What is the sectional curvature of this metric? Is there a more elementary way of describing this metric on S^3?

7. For which values of n does the n-sphere S^n admit a Lie group structure?

[1] As a real vector space, $\mathbb{H}$ is spanned by $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$. Quaternionic multiplication is associative, $\mathbb{R}$-bilinear, and determined by the following rules: $\mathbf{1}$ is the multiplicative identity, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$, $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$. Cyclic permutations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ yield other valid formulæ. The quaternions are denoted by $\mathbb{H}$ in honor of their inventor, W.R.Hamilton.