MAT 552

Introduction to

Lie Groups and Lie Algebras

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Representations of $\mathbf{SU}(2)$

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Here $\odot$ denotes the symmetric tensor product.

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for some $v \neq 0$. This n is called "highest weight".

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$z \in \mathbf{U}(1) \subset \mathbb{C}$ acts with eigenvalues

$$z^{-n}, z^{-n+2}, \ldots, z^{n-2}, z^{n}$$

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These are classified by Dynkin diagrams.