MAT 552

Introduction to

Lie Groups and Lie Algebras

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for any pair of left-invariant vector fields

$$X, Y \in \mathfrak{g}.$$

Curvature Tensor

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For Lie group G with torsion-free bi-invariant ∇

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Sectional Curvature

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Let u, v be orthonormal

$$g(u, u) = g(v, v) = 1, \quad g(u, v) = 0$$

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$$K(\Pi) = g(u, \mathcal{R}_{uv}v)$$

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$$K(\Pi) = \frac{1}{4}\|[X, Y]\|^2$$

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$$\begin{aligned} K(\Pi) &= \frac{1}{4}\|[X, Y]\|^2 \\ &\geq 0. \end{aligned}$$

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Ricci tensor of (M, ∇) defined by

$$\mathsf{Ric}(v, w) := \qquad (u \longmapsto \mathcal{R}_{uv} w)$$

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This makes it enough to know $\mathsf{Ric}(v, v)$.

Riemannian case: $\mathsf{Ric}(v, v)$ for unit vectors v.

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of Lie group G with bi-invariant g:

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of Lie group G with ∇ bi-invariant, torsion-free:

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where Killing form is defined by

$$B(Y, Z) = \mathrm{tr}(Ad_Z \circ Ad_Y) = \mathrm{tr}(Ad_Y \circ Ad_Z)$$