MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

April 6, 2021

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

- $\nabla_v w - \nabla_w v = [v, w]$*; and*

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

- $\nabla_v w - \nabla_w v = [v, w]$; *and*
- $ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$.

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

- $\nabla_v w - \nabla_w v = [v, w]$; *and*
- $ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$.

For any Lie group G with bi-invariant metric g

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

- $\nabla_v w - \nabla_w v = [v, w]$; *and*
- $ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$.

For any Lie group G with bi-invariant metric g

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

- $\nabla_v w - \nabla_w v = [v, w]$*; and*
- $ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$.

For any Lie group G with bi-invariant metric g

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

for any pair of left-invariant vector fields

$$X, Y \in \mathfrak{g}.$$

Curvature Tensor

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\mathcal{R}_{XY}Z = \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z]$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\begin{aligned}\mathcal{R}_{XY}Z &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[Z,X]] + \frac{1}{2}[Z,[X,Y]]\end{aligned}$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\begin{aligned}
\mathcal{R}_{XY}Z &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\
&= \frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[Z,X]] + \frac{1}{2}[Z,[X,Y]] \\
&= \frac{1}{4}\left([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]\right) \\
&\qquad + \frac{1}{4}[Z,[X,Y]]
\end{aligned}$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\begin{aligned}
\mathcal{R}_{XY}Z &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\
&= \frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[Z,X]] + \frac{1}{2}[Z,[X,Y]] \\
&= \frac{1}{4}[Z,[X,Y]]
\end{aligned}$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\mathcal{R}_{XY}Z = \frac{1}{4}[Z,[X,Y]]$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\mathcal{R}_{XY}Z = -\frac{1}{4}[[X,Y],Z]$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For Lie group G with torsion-free bi-invariant ∇

$$\mathcal{R}_{XY}Z = -\frac{1}{4}[[X, Y], Z]$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\mathcal{R}_{XY}Z = -\frac{1}{4}[[X, Y], Z]$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For Lie group G with torsion-free bi-invariant ∇

$$\mathcal{R}_{XY}Z = -\frac{1}{4}[[X,Y],Z]$$

Curvature Tensor

$$\mathcal{R}_{uv}w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

For any Lie group G with bi-invariant metric g

$$\mathcal{R}_{XY}Z = -\frac{1}{4}[[X,Y],Z]$$

Sectional Curvature

Now assume that (M, g) Riemannian.

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal

$$g(u, u) = g(v, v) = 1, \quad g(u, v) = 0$$

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \mathrm{span}(u, v)$.

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \mathrm{span}(u, v)$.

Sectional Curvature in 2-plane Π:

$$K(\Pi) = g(u, \mathcal{R}_{uv} v)$$

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \mathrm{span}(u, v)$.

Sectional Curvature in 2-plane Π:

$$K(\Pi) = g(u, \mathcal{R}_{uv} v)$$

Lie group G with bi-invariant g

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \mathrm{span}(u, v)$.

Sectional Curvature in 2-plane Π:

$$K(\Pi) = g(u, \mathcal{R}_{uv}v)$$

Lie group G with bi-invariant g

$$K(\Pi) = \langle X, -\frac{1}{4}[[X, Y], Y]\rangle$$

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \mathrm{span}(u, v)$.

Sectional Curvature in 2-plane Π:

$$K(\Pi) = g(u, \mathcal{R}_{uv}v)$$

Lie group G with bi-invariant g

$$K(\Pi) = \frac{1}{4}\langle X, [Y, [X, Y]]\rangle$$

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \mathrm{span}(u, v)$.

Sectional Curvature in 2-plane Π:

$$K(\Pi) = g(u, \mathcal{R}_{uv}v)$$

Lie group G with bi-invariant g

$$K(\Pi) = -\frac{1}{4}\langle [Y, X], [X, Y] \rangle$$

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \mathrm{span}(u, v)$.

Sectional Curvature in 2-plane Π:

$$K(\Pi) = g(u, \mathcal{R}_{uv} v)$$

Lie group G with bi-invariant g

$$K(\Pi) = \frac{1}{4}\langle [X, Y], [X, Y] \rangle$$

Sectional Curvature

Now assume that (M, g) Riemannian.

Let u, v be orthonormal, and $\Pi = \text{span}(u, v)$.

Sectional Curvature in 2-plane Π:

$$K(\Pi) = g(u, \mathcal{R}_{uv} v)$$

Lie group G with bi-invariant g

$$K(\Pi) = \frac{1}{4} \| [X, Y] \|^2$$