MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

April 29, 2021

Proposition. *Let* G *be a compact Lie group,*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$*,*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$, *and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

be a real representation of G*. Then there is a positive-definite inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*.*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

be a real representation of G*. Then there is a positive-definite inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*. Any such representation can therefore be viewed as a Lie-group homomorphism*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$, *and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

be a real representation of G. *Then there is a positive-definite inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G. *Any such representation can therefore be viewed as a Lie-group homomorphism*

$$\varrho : \mathsf{G} \to \mathbf{O}(\mathbb{V}, \langle\ ,\ \rangle) \cong \mathbf{O}(n).$$

Definition. *If* G *is a Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$*,*

Definition. *If* G *is a Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

Definition. *If* G *is a Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

is a real representation of G*,*

Definition. *If* G *is a Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

is a real representation of G*, we say that* ϱ *makes* $\mathbb{V}$ *into a (real)* G*-module.*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$,

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation of G.

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation of G*. Then there is a positive-definite Hermitian inner product* $\langle \ , \ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*.*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation of G*. Then there is a positive-definite Hermitian inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*. Any such representation can therefore be viewed as a Lie-group homomorphism*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$, *and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation of G. *Then there is a positive-definite Hermitian inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G. *Any such representation can therefore be viewed as a Lie-group homomorphism*

$$\varrho : \mathsf{G} \to \mathbf{U}(\mathbb{V}, \langle\ ,\ \rangle) \cong \mathbf{U}(n).$$

Definition. *If* G *is a Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and*

Definition. *If* G *is a Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

is a complex representation of G*,*

Definition. *If* G *is a Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

is a complex representation of G*, we say that* ϱ *makes* $\mathbb{V}$ *into a (complex)* G*-module.*

Definition. *A real* G-*module* $\mathbb{V}$ *is irreducible*

Definition. *A real* G*-module* $\mathbb{V}$ *is* *irreducible* *if the only* G*-invariant real vector subspaces*

Definition. *A real* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant real vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

Definition. *A real* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant real vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

are $\mathsf{0}$ *and* $\mathbb{V}$.

Definition. *A complex* G*-module* $\mathbb{V}$ *is irreducible*

Definition. *A complex* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant complex vector subspaces*

Definition. *A complex* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant complex vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

Definition. *A complex* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant complex vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

are $\mathsf{0}$ *and* $\mathbb{V}$.

Theorem. *Let* G *be a compact Lie group.*

Theorem. *Let* G *be a compact Lie group. Then any real* G*-module*

Theorem. *Let* G *be a compact Lie group. Then any real* G*-module is a direct sum of irreducible real* G*-modules.*

Theorem. *Let* G *be a compact Lie group. Then any real* G*-module is a direct sum of irreducible real* G*-modules.*

$$\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_1$$

Theorem. *Let* G *be a compact Lie group. Then any real* G*-module is a direct sum of irreducible real* G*-modules.*

$$\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_1$$

$$\rho = \rho_1 \oplus \cdots \oplus \rho_k$$

Theorem. *Let* G *be a compact Lie group. Then any complex* G*-module is a direct sum of irreducible complex* G*-modules.*

$$\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_1$$

$$\rho = \rho_1 \oplus \cdots \oplus \rho_k$$

Definition. *A connected Lie group* G *is called* *simple* *if*

Definition. *A connected Lie group* G *is called* *simple* *if*

- $\mathfrak{g}$ *is non-Abelian; and*

Definition. *A connected Lie group* G *is called simple if*

- $\mathfrak{g}$ *is non-Abelian; and*
- *the adjoint representation* G *on* $\mathfrak{g}$ *is irreducible.*

Lemma (Schur)**.**

Lemma (Schur)**.** *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be* G*-modules*

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be* G-*modules*

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be irreducible* G-*modules*

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be irreducible* G*-modules*

$$\varrho_1 : \mathsf{G} \to \mathbf{GL}(\mathbb{V})$$

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be irreducible* G-*modules*

$$\varrho_1 : \mathsf{G} \to \mathbf{GL}(\mathbb{V})$$

$$\varrho_2 : \mathsf{G} \to \mathbf{GL}(\mathbb{W})$$

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be irreducible* G-*modules*

$$\varrho_1 : \mathsf{G} \to \mathbf{GL}(\mathbb{V})$$

$$\varrho_2 : \mathsf{G} \to \mathbf{GL}(\mathbb{W})$$

and let $\Phi : \mathbb{V} \to \mathbb{W}$

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be irreducible* G*-modules*

$$\varrho_1 : \mathsf{G} \to \mathbf{GL}(\mathbb{V})$$

$$\varrho_2 : \mathsf{G} \to \mathbf{GL}(\mathbb{W})$$

and let $\Phi : \mathbb{V} \to \mathbb{W}$ *be a homomorphism of* G*-modules:*

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be irreducible* G-*modules*

$$\varrho_1 : \mathsf{G} \to \mathbf{GL}(\mathbb{V})$$

$$\varrho_2 : \mathsf{G} \to \mathbf{GL}(\mathbb{W})$$

and let $\Phi : \mathbb{V} \to \mathbb{W}$ *be a homomorphism of* G-*modules:*

$$\varrho_2(\mathsf{a}) \cdot \Phi(v) = \Phi(\varrho_1(\mathsf{a}) \cdot v).$$

Lemma (Schur). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be irreducible* G*-modules*

$$\varrho_1 : \mathsf{G} \to \mathbf{GL}(\mathbb{V})$$

$$\varrho_2 : \mathsf{G} \to \mathbf{GL}(\mathbb{W})$$

and let $\Phi : \mathbb{V} \to \mathbb{W}$ *be a homomorphism of* G*-modules:*

$$\varrho_2(\mathsf{a}) \cdot \Phi(v) = \Phi(\varrho_1(\mathsf{a}) \cdot v).$$

Then either $\Phi = 0$, *or else* Φ *is an isomorphism.*

Representations of $\mathbf{SU}(2)$

Representations of $\mathbf{SU}(2)$

Theorem.

Representations of $\mathbf{SU}(2)$

Theorem. *The complex irreducible representations of* $\mathbf{SU}(2)$ *are classified by the non-negative integers* n*.*

Representations of $\mathbf{SU}(2)$

Theorem. *The complex irreducible representations of* $\mathbf{SU}(2)$ *are classified by the non-negative integers* n. *For each* n, *there is, up to isomorphism, a unique irreducible* $\mathbf{SU}(2)$*-module* $\mathbb{V}_n$ *of dimension* $n+1$. *This can be expressed in terms*

Representations of $\mathbf{SU}(2)$

Theorem. *The complex irreducible representations of* $\mathbf{SU}(2)$ *are classified by the non-negative integers* n. *For each* n, *there is, up to isomorphism, a unique irreducible* $\mathbf{SU}(2)$*-module* $\mathbb{V}_n$ *of dimension* $n+1$. *This can be expressed in terms of the basic* 2*-dimensional representation* $\mathbb{S}$ *as*

Representations of $\mathbf{SU}(2)$

Theorem. *The complex irreducible representations of* $\mathbf{SU}(2)$ *are classified by the non-negative integers* n*. For each* n*, there is, up to isomorphism, a unique irreducible* $\mathbf{SU}(2)$*-module* $\mathbb{V}_n$ *of dimension* $n+1$*. This can be expressed in terms of the basic* 2*-dimensional representation* $\mathbb{S}$ *as*

$$\mathbb{V}_n = \odot^n \mathbb{S}.$$

Representations of $\mathbf{SU}(2)$

Theorem. *The complex irreducible representations of* $\mathbf{SU}(2)$ *are classified by the non-negative integers* n*. For each* n*, there is, up to isomorphism, a unique irreducible* $\mathbf{SU}(2)$*-module* $\mathbb{V}_n$ *of dimension* $n+1$*. This can be expressed in terms of the basic* 2*-dimensional representation* $\mathbb{S}$ *as*

$$\mathbb{V}_n = \odot^n \mathbb{S}.$$

Here $\odot$ denotes the symmetric tensor product.