

MAT 552

Introduction to

Lie Groups and Lie Algebras

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Stony Brook University

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$$\varrho : \mathbf{G} \rightarrow \mathbf{O}(\mathbb{V}, \langle \cdot, \cdot \rangle) \cong \mathbf{O}(n).$$

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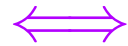
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Then either $\Phi = 0$, or else Φ is an isomorphism.