MAT 552

Introduction to

Lie Groups and Lie Algebras

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Then either $\Phi = 0$, *or else* Φ *is an isomorphism.*