

*MAT 552*

*Introduction to*

*Lie Groups and Lie Algebras*

Claude LeBrun

Stony Brook University

April 22, 2021

**Proposition.** *Let  $G$  be a compact Lie group,*

**Proposition.** *Let  $\mathbf{G}$  be a compact Lie group, let  $\mathbb{V}$  be a vector space over  $\mathbb{R}$ ,*

**Proposition.** *Let  $\mathbf{G}$  be a compact Lie group, let  $\mathbb{V}$  be a vector space over  $\mathbb{R}$ , and let*

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

**Proposition.** *Let  $\mathbf{G}$  be a compact Lie group, let  $\mathbb{V}$  be a vector space over  $\mathbb{R}$ , and let*

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

*be a **real** representation of  $\mathbf{G}$ . Then there is a positive-definite inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{V}$  that is invariant under the action of  $\mathbf{G}$ .*

**Proposition.** *Let  $\mathbf{G}$  be a compact Lie group, let  $\mathbb{V}$  be a vector space over  $\mathbb{R}$ , and let*

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

*be a **real** representation of  $\mathbf{G}$ . Then there is a positive-definite inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{V}$  that is invariant under the action of  $\mathbf{G}$ . Any such representation can therefore be viewed as a Lie-group homomorphism*

**Proposition.** Let  $\mathbf{G}$  be a compact Lie group, let  $\mathbb{V}$  be a vector space over  $\mathbb{R}$ , and let

$$\varrho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

be a *real* representation of  $\mathbf{G}$ . Then there is a positive-definite inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{V}$  that is invariant under the action of  $\mathbf{G}$ . Any such representation can therefore be viewed as a Lie-group homomorphism

$$\varrho : \mathbf{G} \rightarrow \mathbf{O}(\mathbb{V}, \langle \cdot, \cdot \rangle) \cong \mathbf{O}(n).$$

**Definition.** If  $\mathbf{G}$  is a Lie group,  $\mathbb{V}$  is a vector space over  $\mathbb{R}$ ,



**Definition.** If  $\mathbf{G}$  is a Lie group,  $\mathbb{V}$  is a vector space over  $\mathbb{R}$ , and

$$\varrho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

**Definition.** If  $\mathbf{G}$  is a Lie group,  $\mathbb{V}$  is a vector space over  $\mathbb{R}$ , and

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

is a *real* representation of  $\mathbf{G}$ ,

**Definition.** If  $\mathbf{G}$  is a Lie group,  $\mathbb{V}$  is a vector space over  $\mathbb{R}$ , and

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

is a *real* representation of  $\mathbf{G}$ , we say that  $\rho$  makes  $\mathbb{V}$  into a (*real*)  $\mathbf{G}$ -module.

**Proposition.** *Let  $G$  be a compact Lie group, let  $V$  is a vector space over  $\mathbb{C}$ ,*

**Proposition.** *Let  $\mathbf{G}$  be a compact Lie group, let  $\mathbb{V}$  is a vector space over  $\mathbb{C}$ , and let*

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

*be a complex representation of  $\mathbf{G}$ .*

**Proposition.** Let  $G$  be a compact Lie group, let  $V$  is a vector space over  $\mathbb{C}$ , and let

$$\rho : G \rightarrow \mathbf{GL}(V) \cong \mathbf{GL}(n, \mathbb{C})$$

be a *complex* representation of  $G$ . Then there is a positive-definite *Hermitian* inner product  $\langle \cdot, \cdot \rangle$  on  $V$  that is invariant under the action of  $G$ .

**Proposition.** Let  $G$  be a compact Lie group, let  $V$  is a vector space over  $\mathbb{C}$ , and let

$$\rho : G \rightarrow \mathbf{GL}(V) \cong \mathbf{GL}(n, \mathbb{C})$$

be a *complex* representation of  $G$ . Then there is a positive-definite *Hermitian* inner product  $\langle \cdot, \cdot \rangle$  on  $V$  that is invariant under the action of  $G$ . Any such representation can therefore be viewed as a Lie-group homomorphism

**Proposition.** Let  $\mathbf{G}$  be a compact Lie group, let  $\mathbb{V}$  is a vector space over  $\mathbb{C}$ , and let

$$\varrho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a *complex* representation of  $\mathbf{G}$ . Then there is a positive-definite *Hermitian* inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{V}$  that is invariant under the action of  $\mathbf{G}$ . Any such representation can therefore be viewed as a Lie-group homomorphism

$$\varrho : \mathbf{G} \rightarrow \mathbf{U}(\mathbb{V}, \langle \cdot, \cdot \rangle) \cong \mathbf{U}(n).$$



**Definition.** If  $\mathbf{G}$  is a Lie group,  $\mathbb{V}$  is a vector space over  $\mathbb{C}$ , and

**Definition.** If  $\mathbf{G}$  is a Lie group,  $\mathbb{V}$  is a vector space over  $\mathbb{C}$ , and

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

is a *complex* representation of  $\mathbf{G}$ ,

**Definition.** If  $\mathbf{G}$  is a Lie group,  $\mathbb{V}$  is a vector space over  $\mathbb{C}$ , and

$$\rho : \mathbf{G} \rightarrow \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

is a *complex* representation of  $\mathbf{G}$ , we say that  $\rho$  makes  $\mathbb{V}$  into a (*complex*)  $\mathbf{G}$ -module.

**Definition.** A real  $G$ -module  $V$  is *irreducible*

**Definition.** A real  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant real vector subspaces

**Definition.** A real  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant real vector subspaces

$$W \subset V$$

**Definition.** A real  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant real vector subspaces

$$W \subset V$$

are  $0$  and  $V$ .

**Definition.** A real  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant real vector subspaces

$$W \subset V$$

are  $0$  and  $V$ .

**Example.** Usual action of  $\mathbf{SO}(n)$  on  $\mathbb{R}^n$ ,



**Definition.** A real  $\mathbf{G}$ -module  $\mathbb{V}$  is *irreducible* if the only  $\mathbf{G}$ -invariant real vector subspaces

$$\mathbb{W} \subset \mathbb{V}$$

are  $0$  and  $\mathbb{V}$ .

**Example.** Usual action of  $\mathbf{SO}(n)$  on  $\mathbb{R}^n$ ,  
because  $\mathbf{SO}(n)$  acts transitively on  $S^{n-1}$ .

**Definition.** A complex  $G$ -module  $V$  is irreducible

**Definition.** A complex  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant complex vector subspaces

**Definition.** A complex  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant complex vector subspaces

$$W \subset V$$

**Definition.** A complex  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant complex vector subspaces

$$W \subset V$$

are  $0$  and  $V$ .

**Definition.** A complex  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant complex vector subspaces

$$W \subset V$$

are  $0$  and  $V$ .

**Example.** Usual action of  $SU(n)$  on  $\mathbb{C}^n$ ,

**Definition.** A complex  $\mathbf{G}$ -module  $\mathbb{V}$  is *irreducible* if the only  $\mathbf{G}$ -invariant complex vector subspaces

$$\mathbb{W} \subset \mathbb{V}$$

are  $0$  and  $\mathbb{V}$ .

**Example.** Usual action of  $\mathbf{SU}(n)$  on  $\mathbb{C}^n$ ,  
because  $\mathbf{SU}(n)$  acts transitively on  $S^{2n-1}$ .

**Definition.** A complex  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant complex vector subspaces

$$W \subset V$$

are  $0$  and  $V$ .



**Definition.** A complex  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant complex vector subspaces

$$W \subset V$$

are  $0$  and  $V$ .

**Example.** Usual action of  $\mathbf{SO}(n)$  on  $\mathbb{C}^n$ ,  $n \geq 3$ :

**Definition.** A complex  $G$ -module  $V$  is *irreducible* if the only  $G$ -invariant complex vector subspaces

$$W \subset V$$

are  $0$  and  $V$ .

**Example.** Usual action of  $\mathbf{SO}(n)$  on  $\mathbb{C}^n$ ,  $n \geq 3$ :

Irreducible over  $\mathbb{C}$ , but *reducible* over  $\mathbb{R}$ .

**Theorem.** *Let  $G$  be a compact Lie group.*

**Theorem.** *Let  $G$  be a compact Lie group. Then  
a real  $G$ -module  $V$*

**Theorem.** *Let  $G$  be a compact Lie group. Then a real  $G$ -module  $V$  is irreducible*

**Theorem.** Let  $G$  be a compact Lie group. Then  
a real  $G$ -module  $V$  is *irreducible*



**Theorem.** Let  $G$  be a compact Lie group. Then  
a real  $G$ -module  $V$  is *irreducible*



the  $G$ -invariant positive-definite inner product  $\langle \cdot, \cdot \rangle$

**Theorem.** Let  $G$  be a compact Lie group. Then a real  $G$ -module  $V$  is *irreducible*



the  $G$ -invariant positive-definite inner product  $\langle \cdot, \cdot \rangle$  is *unique*



**Theorem.** Let  $G$  be a compact Lie group. Then a real  $G$ -module  $V$  is *irreducible*



the  $G$ -invariant positive-definite inner product  $\langle \cdot, \cdot \rangle$  is *unique* up to scale.

**Theorem.** *Let  $G$  be a compact Lie group.*

**Theorem.** *Let  $G$  be a compact Lie group. Then a complex  $G$ -module  $V$  is irreducible*

**Theorem.** *Let  $G$  be a compact Lie group. Then a complex  $G$ -module  $V$  is irreducible*



**Theorem.** Let  $G$  be a compact Lie group. Then a complex  $G$ -module  $V$  is irreducible



the  $G$ -invariant Hermitian inner product  $\langle \cdot, \cdot \rangle$  is unique

**Theorem.** Let  $G$  be a compact Lie group. Then a complex  $G$ -module  $V$  is irreducible



the  $G$ -invariant Hermitian inner product  $\langle \cdot, \cdot \rangle$  is unique up to scale.

**Theorem.** *Let  $G$  be a compact Lie group.*

**Theorem.** *Let  $G$  be a compact Lie group. Then any real  $G$ -module*



**Theorem.** *Let  $G$  be a compact Lie group. Then any real  $G$ -module is a direct sum of irreducible real  $G$ -modules.*

**Theorem.** Let  $G$  be a compact Lie group. Then any real  $G$ -module is a direct sum of *irreducible* real  $G$ -modules.

$$V = V_1 \oplus \cdots \oplus V_1$$

**Theorem.** Let  $G$  be a compact Lie group. Then any real  $G$ -module is a direct sum of *irreducible* real  $G$ -modules.

$$V = V_1 \oplus \cdots \oplus V_1$$

$$\rho = \rho_1 \oplus \cdots \oplus \rho_k$$

**Theorem.** Let  $G$  be a compact Lie group.  
Then any complex  $G$ -module is a direct sum of  
irreducible complex  $G$ -modules.

$$V = V_1 \oplus \cdots \oplus V_1$$

$$\rho = \rho_1 \oplus \cdots \oplus \rho_k$$

**Definition.** A connected Lie group  $G$  is called *simple* if

**Definition.** A connected Lie group  $G$  is called simple if

- $\mathfrak{g}$  is non-Abelian; and

**Definition.** A connected Lie group  $G$  is called *simple* if

- $\mathfrak{g}$  is non-Abelian; and
- the adjoint representation  $G$  on  $\mathfrak{g}$  is *irreducible*.