MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

April 20, 2021

Proposition. *Let* G *be a compact Lie group,*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$*,*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

be a real representation on G*. Then there is a positive-definite inner product* $\langle \ , \ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*.*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

be a ***real*** *representation on* G*. Then there is a positive-definite inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*. Any such representation can therefore be viewed as a Lie-group homomorphism*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *be a vector space over* $\mathbb{R}$, *and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

be a real representation on G. *Then there is a positive-definite inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G. *Any such representation can therefore be viewed as a Lie-group homomorphism*

$$\varrho : \mathsf{G} \to \mathbf{O}(\mathbb{V}, \langle\ ,\ \rangle) \cong \mathbf{O}(n).$$

Definition. *If* G *is a compact Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$*,*

Definition. *If* G *is a compact Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$, *and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

Definition. *If* G *is a compact Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$, *and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

is a real representation on G,

Definition. *If* G *is a compact Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{R})$$

is a real representation on G*, we say that* ϱ *makes* $\mathbb{V}$ *into a (real)* G*-module.*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*,*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation on G.

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation on G*. Then there is a positive-definite Hermitian inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*.*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation on G*. Then there is a positive-definite Hermitian inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*. Any such representation can therefore be viewed as a Lie-group homomorphism*

Proposition. *Let* G *be a compact Lie group, let* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and let*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

be a complex representation on G*. Then there is a positive-definite Hermitian inner product* $\langle\ ,\ \rangle$ *on* $\mathbb{V}$ *that is invariant under the action of* G*. Any such representation can therefore be viewed as a Lie-group homomorphism*

$$\varrho : \mathsf{G} \to \mathbf{U}(\mathbb{V}, \langle\ ,\ \rangle) \cong \mathbf{U}(n).$$

Definition. *If* G *is a compact Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and*

Definition. *If* G *is a compact Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

is a complex representation on G*,*

Definition. *If* G *is a compact Lie group,* $\mathbb{V}$ *is a vector space over* $\mathbb{C}$*, and*

$$\varrho : \mathsf{G} \to \mathbf{GL}(\mathbb{V}) \cong \mathbf{GL}(n, \mathbb{C})$$

is a complex representation on G*, we say that* ϱ *makes* $\mathbb{V}$ *into a (complex)* G*-module.*

Definition. *A real* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant real vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

are $\mathsf{0}$ *and* $\mathbb{V}$*.*

Definition. *A real* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant real vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

are $\mathsf{0}$ *and* $\mathbb{V}$*.*

Example. Usual action of $\mathbf{SO}(n)$ on $\mathbb{R}^n$,

Definition. *A real* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant real vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

are $\mathsf{0}$ *and* $\mathbb{V}$*.*

Example. Usual action of $\mathbf{SO}(n)$ on $\mathbb{R}^n$,

because $\mathbf{SO}(n)$ acts transitively on S^{n-1}.

Definition. *A complex* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant complex vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

are $\mathsf{0}$ *and* $\mathbb{V}$*.*

Example. Usual action of $\mathbf{SU}(n)$ on $\mathbb{C}^n$,

because $\mathbf{SU}(n)$ acts transitively on S^{2n-1}.

Definition. *A complex* G*-module* $\mathbb{V}$ *is irreducible if the only* G*-invariant complex vector subspaces*

$$\mathbb{W} \subset \mathbb{V}$$

are $\mathsf{0}$ *and* $\mathbb{V}$.

Example. Usual action of $\mathbf{SO}(n)$ on $\mathbb{C}^n$:

Irreducible over $\mathbb{C}$, but reducible over $\mathbb{R}$.

Theorem. *Let* G *be a compact Lie group. Then any (real /complex)* G*-module is a direct sum of irreducible (real/complex)* G*-modules.*