MAT 552

Introduction to

Lie Groups and Lie Algebras

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Recall, Killing form is defined by

$$B(X, Y) = \mathrm{tr}(Ad_X \circ Ad_Y)$$

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Corollary. *If* G *is any compact Lie group for which* $\mathfrak{z}(\mathfrak{g}) = \mathsf{0}$, *then the fundamental group* $\pi_1(\mathsf{G})$ *is finite.*

Theorem. *If* G *is any compact Lie group, then* G *is finitely covered by*

$$\mathbb{T}^k \times \widetilde{\mathsf{G}}$$

for some $k \geq 0$, *where* $\widetilde{\mathsf{G}}$ *is a compact simply-connected Lie group with Lie algebra* $\widetilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}$.

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of Lie group G with ∇ bi-invariant, torsion-free:

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for some constant $C > 0$. *Then* M *is compact.*

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$\Longleftarrow$: Take $g = -B$.