MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

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for any pair of left-invariant vector fields

$$X, Y \in \mathfrak{g}.$$

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$$g(u, u) = g(v, v) = 1, \quad g(u, v) = 0$$

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$$\begin{aligned} K(\Pi) &= \frac{1}{4}\|[X, Y]\|^2 \\ &\geq 0. \end{aligned}$$

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This makes it enough to know $\mathsf{Ric}(v, v)$.

Riemannian case: $\mathsf{Ric}(v, v)$ for unit vectors v.

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$\Longleftarrow$: Take $g = -B$.

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Corollary. *If* G *is any compact Lie group for which* $\mathfrak{z}(\mathfrak{g}) = \mathsf{0}$, *then the fundamental group* $\pi_1(\mathsf{G})$ *is finite.*