MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

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which defines a positive-definite inner product on each tangent space $T_p M$:

$$v \neq 0 \implies g(v, v) > 0.$$

For any piece-wise smooth path

$$\gamma : [a, b] \to M$$

in (M, g) we define its length to be

$$L(\gamma) = \int_a^b |\gamma'(t)|_g \, dt.$$

We say that γ is a path from p to q if

$$\gamma(a) = p \quad \text{and} \quad \gamma(b) = q.$$

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Proposition. *This definition makes* (M, dist) *into a metric space.*

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

- $\nabla_v w - \nabla_w v = [v, w]$; *and*
- $ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$.

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for every $t \in (a, b)$, *where* ∇ *denotes the Riemannian connection determined by* g.

Example. Let G be a compact Lie group, and let g be a bi-invariant metric.

Let X be a left-invariant vector field on G.

Then $\nabla_X X = \frac{1}{2}[X, X] = 0$, since

$$\nabla_X Y = \frac{1}{2}[X, Y] \quad \forall\, X, Y \in \mathfrak{g}.$$

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In particular, the curves

$$t \mapsto \exp(tX)$$

are exactly the geodesics through e.

These are the "one-parameter subgroups" of G.

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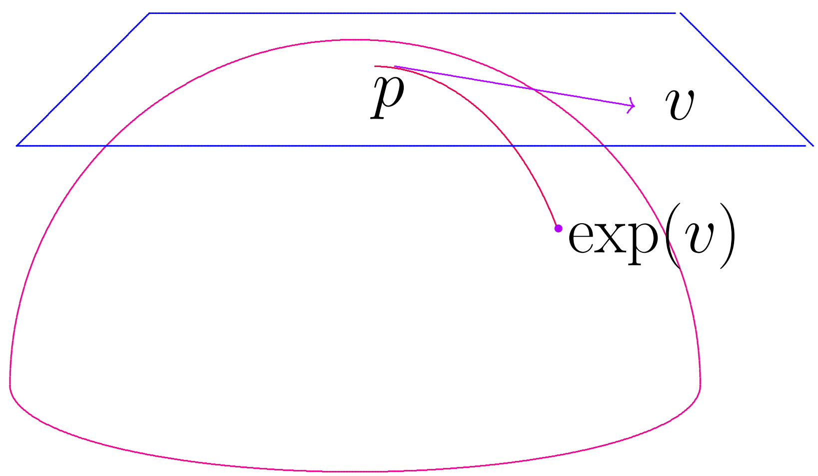
which is at least defined in a neighborhood of 0:



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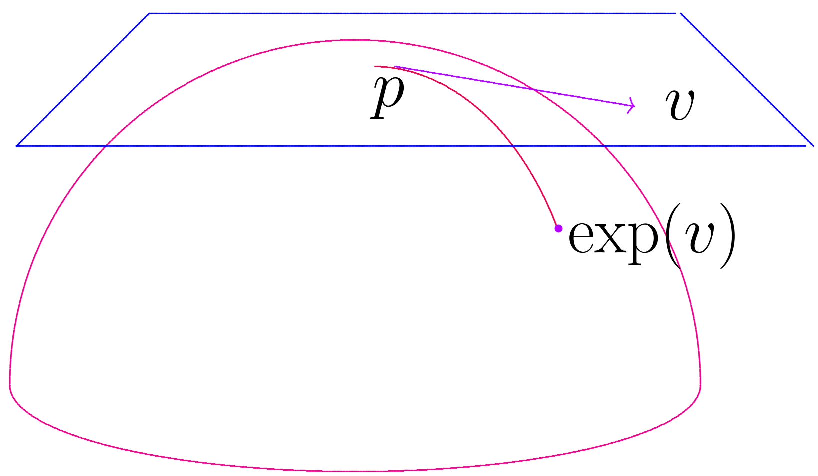

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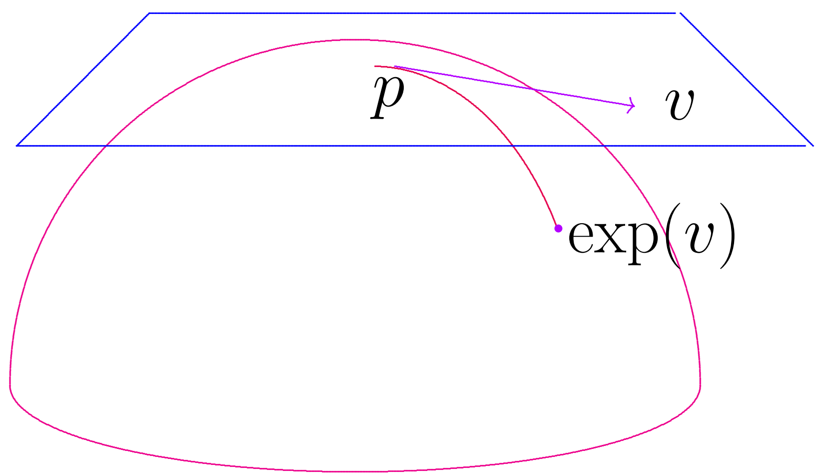

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For G with bi-invariant g, equals Lie-theoretic exp.

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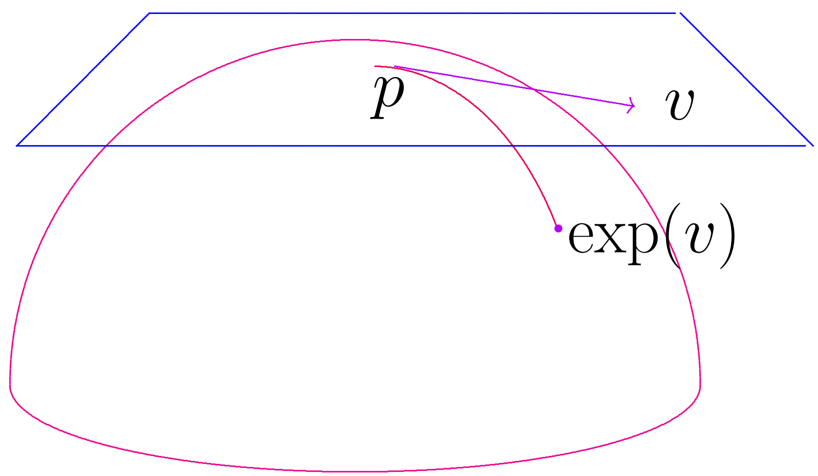

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