# Homework # 3

MAT 552

Optional Problems for Further Explorations.

These problems will not be graded, or even collected!

1. The compact Lie group $\mathbf{SU}(2)$ acts on its own Lie algebra $\mathfrak{su}(2) \cong (\mathbb{R}^3, \times)$ via the adjoint action. Show that the transformations that arise in this way can all be viewed as orthogonal transformations of $\mathbb{R}^3$. Use this to give a different proof of the fact that $\mathbf{SU}(2) \cong \mathbf{Sp}(1)$ is the universal cover of $\mathbf{SO}(3)$.

2. Recall that we can embed $\mathbf{U}(1) \approx S^1$ as a Lie subgroup of $\mathbf{SU}(2)$ by

$$\zeta \mapsto \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} \qquad \forall \zeta \in \mathbf{U}(1) \subset \mathbb{C} - \{0\}.$$

(a) Use problem 1 to show that any other Lie-group embedding

$$\mathbf{U}(1) \hookrightarrow \mathbf{SU}(2)$$

is obtained from the above embedding by conjugation in $\mathbf{SU}(2)$.

(b) Show that it is impossible to embed $\mathbf{U}(1) \times \mathbf{U}(1)$ in $\mathbf{SU}(2)$ as a Lie subgroup. Thus, $\mathbf{U}(1)$ is the maximal torus of $\mathbf{SU}(2)$, in the sense that it is the highest-dimensional Lie subgroup that is a product of circles $\mathbf{U}(1)$.

(c) Observe that $\mathbb{T}^2 \approx \mathbf{U}(1) \times \mathbf{U}(1)$ can nonetheless be embedded in $\mathbf{SU}(2)$ as a submanifold. Thus, the maximal torus only has maximal dimension among compact Abelian Lie subgroups, not among embedded submanifolds.

3. Let $\mathsf{G}$ be a compact connected Abelian Lie group of real dimension $n$. Show that the exponential map from $\mathfrak{g} \cong \mathbb{R}^n$ to $\mathsf{G}$ is both a Lie group homomorphism and a covering map, and that its kernel is a lattice $\Lambda \subset \mathbb{R}^n$, meaning a discrete additive subgroup that spans $\mathbb{R}^n$. Then show any lattice $\Lambda$ in $\mathbb{R}^n$ is exactly the subgroup generated over $\mathbb{Z}$ by some basis for $\mathbb{R}^n$. Then use this to show that $\mathsf{G}$ is isomorphic to the $n$-torus

$$\mathbb{T}^n = \underbrace{\mathbf{U}(1) \times \cdots \times \mathbf{U}(1)}_{n}$$

as a Lie group. As a corollary, deduce that the quotient of $\mathbb{T}^n$ by any finite subgroup is again isomorphic to $\mathbb{T}^n$.

4. Let $\mathsf{G}$ be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$ be any element, let $\exp(\mathbb{R}X)$ be the image of the span of $X$ under the exponential map, and let $\overline{\exp(\mathbb{R}X)}$ denote its closure. Prove that $\overline{\exp(\mathbb{R}X)}$ is a compact connected Abelian subgroup of $\mathsf{G}$.

It is a highly non-trivial theorem[1] that any closed subgroup of a Lie group is a Lie group. Using this fact and Problem 2, conclude that $\overline{\exp \mathbb{R}X}$ must be isomorphic to a torus $\mathbb{T}^n$ for some $n$.

Conversely, if $\mathsf{G}$ contains a Lie subgroup isomorphic to $\mathbb{T}^n$ for some $n$, show that this subgroup arises as $\overline{\exp \mathbb{R}X}$ for some $X \in \mathfrak{g}$.

5. Let $\rho : \mathbf{SU}(2) \to \mathrm{End}(\mathbb{V})$ be a complex representation. Recall that we can then decompose $\mathbb{V}$ into 1-complex-dimensional subspaces, each of which is an eigenspace for the action of $\mathbf{U}(1) \subset \mathbf{SU}(2)$, and that we say that such an eigenspace has weight $n$ if every $\zeta \in \mathbf{U}(1)$ acts on it with eigenvalue $\zeta^n$. The highest weight of the given representation is then the largest integer $n$ that occurs as the weight of some (non-zero) eigenspace.

If $\dim_{\mathbb{C}} \mathbb{V} = n + 1$, show that $\rho$ is an irreducible complex representation if only if it is has highest weight $n$.

If $\mathbb{S} = \mathbb{C}^2$ is the "defining" representation of $\mathbf{SU}(2)$, show that the induced action on the symmetric product $\odot^n \mathbb{S}$ has weight $n$. Also check that $\odot^n \mathbb{S}$ has dimension $n+1$ by writing down as basis. Conclude that $\odot^n \mathbb{S}$ is actually an irreducible representation of $\mathbf{SU}(2)$.

[1] See, for example, Duistermaat and Kolk, Lie Groups, Corollary 1.10.7.

6. Show that the maximal torus of $\mathbf{SO}(3)$ is the image of the obvious inclusion

$$\begin{array}{ccc} \mathbf{SO}(2) & \hookrightarrow & \mathbf{SO}(3) \\ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} & \mapsto & \begin{bmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix} \end{array}$$

and that this circle is double-covered by the maximal torus $\mathbf{U}(1)$ of $\mathbf{SU}(2)$.

One may classify irreducible complex representations of $\mathbf{SO}(3)$ by breaking them into eigenspaces of the action of $\mathbf{SO}(2)$, and then extracting an integer, called the spin, which is the "largest eigenvalue" of all these circle actions. This definition of the spin of an $\mathbf{SO}(3)$-representation was introduced by physicists in the early days of quantum mechanics, and is exactly *twice* the highest weight for the corresponding representation of $\mathbf{SU}(2)$. This led to the perplexing discovery that there are also "half-integer spin" representations of the Lie algebra $\mathfrak{so}(3)$, which actually arise from representations of the universal cover $\mathbf{SU}(2)$ of $\mathbf{SO}(3)$. Physicists therefore still describe irreducible representations of $\mathbf{SU}(2)$ by saying that the one of highest weight $n$ has spin $n/2$. This terminology also helps explain why the universal cover of $\mathbf{SO}(m)$, $m \geq 3$, is known as $\mathbf{Spin}(m)$, even in pure mathematics.

Let $\mathsf{V} = \mathbb{R}^3$ be the tautological real representation of $\mathbf{SO}(3)$, and let $\mathsf{V}_{\mathbb{C}} = \mathbb{C}^3$ be its complexification. Let $\odot_0^m \mathsf{V}^*$ be the totally-symmetric trace-free tensors on $\mathsf{V}$, so that an element $\phi$ is a multi-linear map satisfying

$$\phi(v_1, v_2, \ldots, v_m) = \phi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(m)}) \quad \forall \sigma \in \mathfrak{S}_m, \; v_j \in \mathsf{V},$$

and

$$\sum_{i=1}^{3} \phi(e_i, e_i, v_3, \ldots, v_m) = 0 \quad \forall v_3, \ldots, v_m \in \mathsf{V},$$

where $e_i$ is the usual orthonormal basis for $\mathsf{V} = \mathbb{R}^3$. Prove that this space has dimension $2m + 1$, and that its complexification $\odot^m \mathsf{V}_{\mathbb{C}}$ has spin $m$ as an $\mathbf{SO}(3)$ representation. Since $\odot^m \mathsf{V}_{\mathbb{C}}$ therefore has highest weight $2m$ as an $\mathbf{SU}(2)$ representation, and since its dimension is $2m + 1$, conclude that it is irreducible as a complex representation of $\mathbf{SU}(2)$. Also observe that while this this complex representation is certainly reducible as a real representation, because it can be written as $[\odot_0^m \mathsf{V}] \oplus i[\odot_0^m \mathsf{V}^*]$, the real representation $\odot_0^m \mathsf{V}$ is necessarily irreducible as a real representation.

7. The universal cover $\mathbf{Spin}(4)$ of $\mathbf{SO}(4)$ is isomorphic to $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$, because there is a natural double cover $\mathbf{Sp}(1) \times \mathbf{Sp}(1) \to \mathbf{SO}(4)$ that arises by letting a pair of unit-norm quaternions $(q_1, q_2) \in \mathbf{Sp}(1) \times \mathbf{Sp}(1)$ act on quaternions $\mathsf{a} \in \mathbb{H} = \mathbb{R}^4$ by

$$\mathsf{a} \mapsto q_1 \mathsf{a} q_2^{-1}.$$

Show that the maximal torus of $\mathbf{SO}(4)$ is exactly the image of the obvious embedding $\mathbf{SO}(2) \times \mathbf{SO} \hookrightarrow \mathbf{SO}(4)$, but that this 2-torus is double-covered by the maximal torus $\mathbf{U}(1) \times \mathbf{U}(1)$ of $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$. Show that this implies that complex irreducible representations of $\mathbf{Spin}(4) = \mathbf{Sp}(1) \times \mathbf{Sp}(1)$ are classified (up to isomorphism) by pairs $(m, n)$ of non-negative integers, and that such a representation arises from an $\mathbf{SO}(4)$-representation if and only if $m \equiv n \bmod 2$. The pair $(m, n)$ associated with an irreducible representation is called the representation's *highest weight*.

8. Let $\mathbb{V} = \mathbb{C}^4$ be the "defining" representation of $\mathbf{SU}(4)$, and consider the induced actions of $\mathbf{SU}(4)$ on $\Lambda^2\mathbb{V} \cong \mathbb{C}^6$ and $\Lambda^4\mathbb{V} \cong \mathbb{C}$. Since the latter is trivial, show that the wedge product $\Lambda^2\mathbb{V} \times \Lambda^2\mathbb{V} \to \Lambda^4\mathbb{V}$ defines a non-degenerate symmetric complex-bilinear inner product that is invariant under $\mathbf{SU}(4)$. By diagonalizing this complex inner product with respect to an invariant Hermitian inner product, show that there is an $\mathbf{SU}(4)$-invariant real subspace $\mathsf{W} \subset \Lambda^2\mathbb{V}$ such that $\Lambda^2\mathbb{V} = \mathsf{W} \oplus i\mathsf{W}$. Now use this to prove that $\mathbf{SU}(4)$ is a double cover of $\mathbf{SO}(6)$, and is therefore isomorphic to its universal cover $\mathbf{Spin}(6)$. Finally, carefully examine the Dynkin diagrams $A_3$ and $B_3$ of $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$, and observe that these are actually the same.

9. Recall that $\mathbf{Sp}(n)$ is the exactly the subgroup of $\mathbf{SU}(2n)$ that preserves a specific complex-symplectic form $\Omega \in \Lambda^2(\mathbb{C}^{2n})^*$. Using this, show that when the action off $\mathbf{SU}(4)$ on $\Lambda^2\mathbb{V} \cong \mathbb{C}^6$ is restricted to $\mathbf{Sp}(2)$, it becomes reducible: there is a direct-sum decomposition $\Lambda^2\mathbb{V} \cong \mathbb{C} \oplus \mathbb{C}^5$ that is invariant under $\mathbf{Sp}(2)$. Then use this to show that there is an induced invariant direct-sum decomposition $\mathsf{W} \cong \mathbb{R} \oplus \mathbb{R}^5$. Then use this to prove that $\mathbf{Sp}(2)$ is a double cover of $\mathbf{SO}(5)$, and is therefore isomorphic to its universal cover $\mathbf{Spin}(5)$. Finally, carefully examine the Dynkin diagrams $B_2$ and $C_2$ of $\mathfrak{so}(5)$ and $\mathfrak{sp}(2)$, and observe that these are actually the same.

10. Explicitly describe the maximal tori of $\mathbf{SO}(5)$, $\mathbf{SO}(6)$, $\mathbf{SU}(3)$, $\mathbf{SU}(4)$, and $\mathbf{Sp}(2)$. Then explictly find the weights of their adjoint representations, and explain how these give rise to the corresponding Dynkin diagrams.