MAT 552

Introduction to

Lie Groups and Lie Algebras

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- *bi-invariant if it is* both *left- and right-invariant.*

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Proof. One can carry any given tensor at e to any $\mathsf{a} \in \mathsf{G}$ by $L_{\mathsf{a}} : T_{\mathsf{e}}\mathsf{G} \to T_a\mathsf{G}$.

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which are invariant under the adjoint action of G *on* $T_{\mathsf{e}}\mathsf{G}$*:*

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$$Ad^*_{\mathsf{a}}\varphi = \varphi \quad \forall \mathsf{a} \in \mathsf{G}.$$

$$Ad^*_{\mathsf{a}}\varphi = R^*_{\mathsf{a}^{-1}} L^*_{\mathsf{a}}\varphi$$

Theorem. *Let* G *be any compact connected Lie group of real dimension* n. *Then the space of bi-invariant* n*-forms* $\mu \in \Omega^n(\mathsf{G})$ *on* G *is one-dimensional. Moreover, there is a unique bi-invariant* n*-form* $\mu \in \Omega^n(\mathsf{G})$ *such that*

$$\int_{\mathsf{G}} \mu = 1.$$

Theorem. *Let* G *be any compact Lie group. Then* G *carries a unique bi-invariant smooth measure* $|\mu|$ *such that*

$$\int_{\mathsf{G}} |\mu| = 1.$$

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because $R_{\mathsf{b}}{}^*|\mu| = |\mu|$.

Average vector-valued function $f : \mathsf{G} \to \mathbb{R}^n$ by

$$\overline{f} := \int_{\mathsf{G}} f|\mu| \in \mathbb{R}^n$$

where integral taken component-by-component, so

$$\overline{(f_1, \ldots, f_n)} = (\overline{f_1}, \ldots, \overline{f_n}).$$

If $f : \mathsf{G} \to \mathbb{V}$ is a continuous vector-space-valued function, can similarly define its average by

$$\overline{f} = \int_{\mathsf{G}} f |\mu|$$

and this coincides with the $\mathbb{R}^n$-valued case relative to any basis for $\mathbb{V}$.

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is said to be a representation of G *on* $\mathbb{V}$.

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Then

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$$\overline{\varphi} = \int_{a \in G} \varrho(a)(\varphi)\ |\mu|_a$$

$$\begin{aligned} \varrho(b)(\overline{\varphi}) &= \varrho(b)\left(\int_{a \in G} \varrho(a)(\varphi)\ |\mu|_a\right) \\ &= \int_{a \in G} \varrho(b)\varrho(a)(\varphi)\ |\mu|_a \end{aligned}$$

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be a representation on G. *Let* $\varphi \in \mathbb{V}$, *and set*

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Warning: This process often yields $\overline{\varphi} = 0$!

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