MAT 552

Introduction to

Lie Groups and Lie Algebras

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- *bi-invariant if it is* both *left- and right-invariant.*

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Proof. One can carry any given tensor at e to any $\mathsf{a} \in \mathsf{G}$ by $L_{\mathsf{a}} : T_{\mathsf{e}}\mathsf{G} \to T_a\mathsf{G}$.

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$$Ad^*_{\mathsf{a}}\varphi = R^*_{\mathsf{a}^{-1}} L^*_{\mathsf{a}}\varphi$$

Theorem. *Let* G *be any compact connected Lie group of real dimension* n*. Then the space of bi-invariant* n*-forms* $\mu \in \Omega^n(\mathsf{G})$ *on* G *is one-dimensional. Moreover, there is a unique bi-invariant* n*-form* $\mu \in \Omega^n(\mathsf{G})$ *such that*

$$\int_{\mathsf{G}} \mu = 1.$$

Theorem. *Let* G *be any compact Lie group. Then* G *carries a unique bi-invariant smooth measure* $|\mu|$ *such that*

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Our next objective:

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and we get a representation $Ad_* : \mathsf{G} \to \mathbf{GL}(T_{\mathsf{e}}\mathsf{G})$ called the adjoint representation.

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To find out, we'll use the **Lie derivative**.

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When V not complete, "flow" only defined on a neighborhood of $M \times \{0\} \subset M \times \mathbb{R}$:

$$\Phi : M \times \mathbb{R} \dashrightarrow M$$

where dashed arrow means "not defined everywhere."

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Lie derivative of tensor field φ w/resp. to V:

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where Φ_t is the flow of V.

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Lie bracket:

$$[V, W]f = V(Wf) - W(Vf).$$

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