MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

March 30, 2021

Definition. *A Riemannian metric g on a smooth manifold M is a smooth symmetric tensor field*

$$g \in C^{\infty}(\odot^2 T^* M)$$

Definition. *A Riemannian metric* g *on a smooth manifold* M *is a smooth symmetric tensor field*

$$g \in C^\infty(\odot^2 T^*M)$$

which defines a positive-definite inner product on each tangent space T_pM*:*

Definition. *A Riemannian metric* g *on a smooth manifold* M *is a smooth symmetric tensor field*

$$g \in C^\infty(\odot^2 T^* M)$$

which defines a positive-definite inner product on each tangent space $T_p M$*:*

$$v \neq 0 \implies g(v, v) > 0.$$

For any piece-wise smooth path

$$\gamma : [a, b] \rightarrow M$$

in (M, g) we define its length to be

$$L(\gamma) = \int_a^b \sqrt{g\left(\frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt}\right)}dt$$

For any piece-wise smooth path

$$\gamma : [a, b] \to M$$

in (M, g) we define its length to be

$$L(\gamma) = \int_a^b |\gamma'(t)|_g \, dt$$

For any piece-wise smooth path

$$\gamma : [a, b] \to M$$

in (M, g) we define its length to be

$$L(\gamma) = \int_a^b |\gamma'(t)|_g \, dt.$$

We say that γ is a path from p to q if

$$\gamma(a) = p \quad \text{and} \quad \gamma(b) = q.$$

Definition. *Let* (M, g) *be a connected Riemannian manifold.*

Definition. *Let* (M, g) *be a connected Riemannian manifold. For any* $p, q \in M$ *we define*

Definition. *Let* (M, g) *be a connected Riemannian manifold. For any* $p, q \in M$ *we define*

$$\mathrm{dist}(p, q) = \inf\{L(\gamma) \mid \gamma \textit{ piece-wise smooth path from } p \textit{ to } q\}$$

Definition. *Let* (M, g) *be a connected Riemannian manifold. For any* $p, q \in M$ *we define*

$$\mathrm{dist}(p, q) = \inf\{L(\gamma) \mid \gamma \textit{ piece-wise smooth path from } p \textit{ to } q\}$$

Proposition. *This definition makes* (M, dist) *into a metric space.*

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

- $\nabla_v w - \nabla_w v = [v, w]$; *and*
- $u g(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$.

Definition. *A parameterized curve*

Definition. *A parameterized curve*

$$\gamma : (a, b) \to M$$

Definition. *A parameterized curve*

$$\gamma : (a, b) \to M$$

in a Riemannian manifold (M, g)

Definition. *A parameterized curve*

$$\gamma : (a, b) \to M$$

in a Riemannian manifold (M, g) *is said to be a geodesic if*

Definition. *A parameterized curve*

$$\gamma : (a, b) \to M$$

in a Riemannian manifold (M, g) *is said to be a geodesic if*

$$\nabla_{\gamma'}\gamma' = 0$$

Definition. *A parameterized curve*

$$\gamma : (a, b) \to M$$

in a Riemannian manifold (M, g) *is said to be a geodesic if*

$$\nabla_{\gamma'}\gamma' = 0$$

for every $t \in (a, b)$, *where* ∇ *denotes the Riemannian connection determined by* g.

Let (M, g) be a Riemannian n-manifold, $p \in M$.

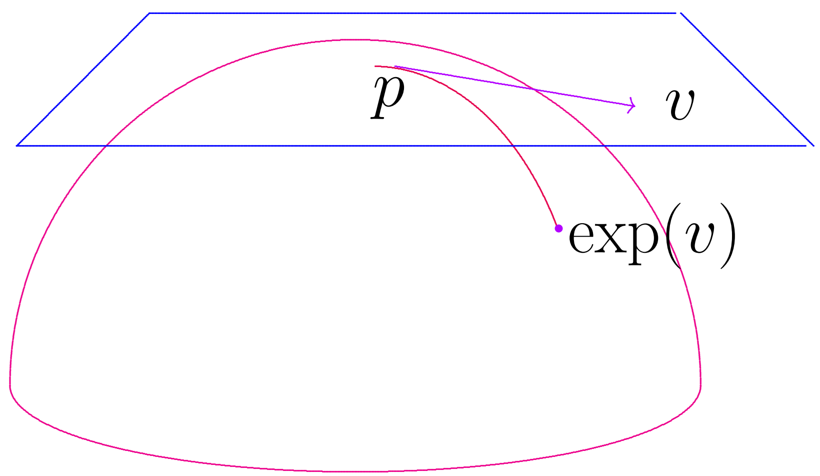
Let (M, g) be a Riemannian n-manifold, $p \in M$.
Metric defines preferred curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \dashrightarrow M$$

Let (M, g) be a Riemannian n-manifold, $p \in M$.
Metric defines preferred curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \dashrightarrow M$$

which is at least defined in a neighborhood of 0:



Definition. *A Riemannian manifold* (M, g) *is called* *geodesically complete at* $p \in M$ *if*

$$\exp : T_pM \to M$$

is defined on the entire tangent space.

Definition. *A Riemannian manifold* (M, g) *is called* *geodesically complete at* $p \in M$ *if*

$$\exp : T_pM \to M$$

is defined on the entire tangent space.

Definition. *A Riemannian manifold* (M, g) *is called* *geodesically complete* *if*

$$\exp : TM \to M$$

is defined on the entire tangent bundle.

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*
- (M, g) *is a geodesically complete at some* p*.*

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*
- (M, g) *is a geodesically complete at some* p*.*

Moreover, any of these conditions implies

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*
- (M, g) *is a geodesically complete at some* p*.*

Moreover, any of these conditions implies

- *every pair of points* $p, q \in M$

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*
- (M, g) *is a geodesically complete at some* p.

Moreover, any of these conditions implies

- *every pair of points* $p, q \in M$ *is joined by a geodesic segment*

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*
- (M, g) *is a geodesically complete at some* p*.*

Moreover, any of these conditions implies

- *every pair of points* $p, q \in M$ *is joined by a geodesic segment*

$$\gamma : [a, b] \to M$$

Theorem (Hopf-Rinow). *For (M, g) connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*
- (M, g) *is a geodesically complete at some* p.

Moreover, any of these conditions implies

- *every pair of points* $p, q \in M$ *is joined by a geodesic segment*

  $$\gamma : [a, b] \to M$$

  of length $L(\gamma) = \mathrm{dist}(p, q)$.

Lemma (Gauss Lemma). *Let* (M, g) *be a connected Riemannian manifold, and let* $p \in M$*. Then, for any sufficiently small* $\varepsilon > 0$, $\exp$ *is a diffeomorphism between* $B_\varepsilon(0) \subset T_pM$ *and* $B_\varepsilon(p) \subset M$*, and the radial geodesic segment from* p *to* q *realizes* $\mathrm{dist}(p, q)$*, and, up to reparameterization, is the unique curve in* M *with this property.*

Lemma (Geodesically Convex Neighborhoods). *Let* (M, g) *be a connected Riemannian manifold, and let* $p \in M$*. Then, for any sufficiently small* $\varepsilon > 0$, $B_\varepsilon(p) \subset M$ *is geodesically convex, in the sense that:*

For any two points $q, r \in B_\varepsilon(p)$*, there is a unique geodesic segment in* $B_\varepsilon(p)$ *joining* q *to* r*. Moreover, this segment realizes* $\operatorname{dist}(q, r)$*, and, up to reparameterization, is the unique curve in* M *with this property.*

Corollary. *Let* M *be a compact manifold,* g *any Riemannian metric on* M*.*

Corollary. *Let* M *be a compact manifold,* g *any Riemannian metric on* M. *Then,* $\forall\ p \in M$,

Corollary. *Let M be a compact manifold, g any Riemannian metric on M. Then,* $\forall\, p \in M$,

$$\exp : T_pM \to M$$

Corollary. *Let M be a compact manifold, g any Riemannian metric on M. Then,* $\forall\ p \in M$,

$$\exp : T_pM \to M$$

is surjective.

Corollary. *Let* G *be a compact Lie group. Then*

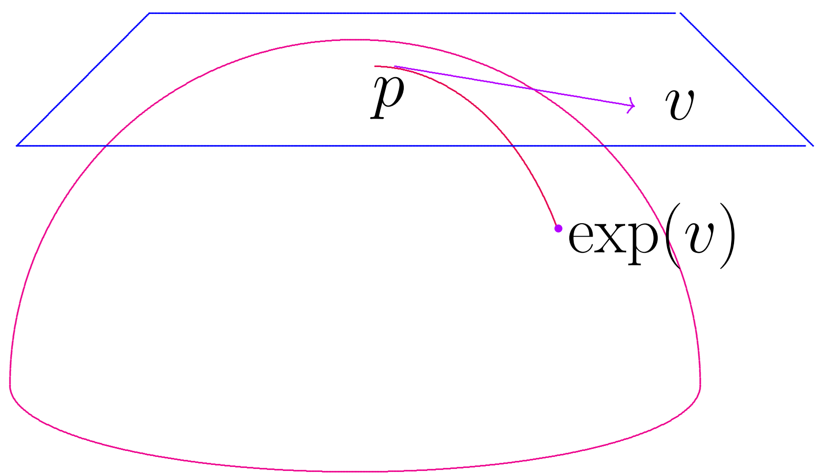
$$\exp : \mathfrak{g} \to \mathsf{G}$$

is surjective.

Let (M, g) be a Riemannian n-manifold, $p \in M$.
Metric defines preferred curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \dashrightarrow M$$

which is at least defined in a neighborhood of 0:



This is a diffeomorphism on a neighborhhood of 0.

For G with bi-invariant g, equals Lie-theoretic exp.

Theorem (Hopf-Rinow). *For* (M, g) *connected Riemannian manifold, the following are equivalent:*

- (M, dist) *is a complete metric space;*
- (M, g) *is a geodesically complete;*
- (M, g) *is a geodesically complete at some* p.

Moreover, any of these conditions implies

- *every pair of points* $p, q \in M$ *is joined by a geodesic segment*

$$\gamma : [a, b] \to M$$

  of length $L(\gamma) = \mathrm{dist}(p, q)$.

Corollary. *Let* G *be a compact Lie group. Then*

$$\exp : \mathfrak{g} \to \mathsf{G}$$

is surjective.

Corollary. *Let* G *be a compact Lie group. Then*

$$\exp : \mathfrak{g} \to \mathsf{G}$$

is surjective.

Not true for non-compact Lie groups!

Corollary. *Let* G *be a compact Lie group. Then*

$$\exp : \mathfrak{g} \to \mathsf{G}$$

is surjective.

Not true for non-compact Lie groups!

We'll see it's not even true for $\mathbf{SL}(2, \mathbb{R})$!