MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

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$$v \neq 0 \implies g(v, v) > 0.$$

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We say that γ is a path from p to q if

$$\gamma(a) = p \quad \text{and} \quad \gamma(b) = q.$$

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$$\mathrm{dist}(p,q) = 0 \quad \Longleftrightarrow \quad p = q$$

Theorem. *Let g be a Riemannian metric on M. Then M admits a unique affine connection ∇ such that*

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for every $t \in (a, b)$*, where* ∇ *denotes the Riemannian connection determined by* g*.*

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These are the "one-parameter subgroups" of G.

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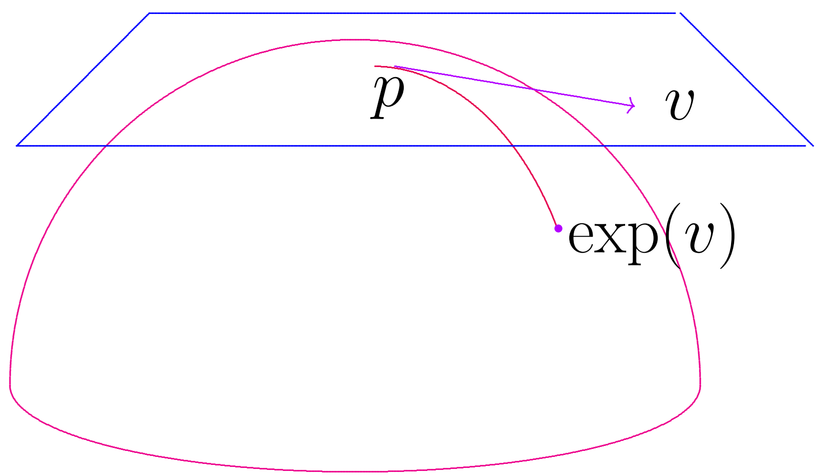

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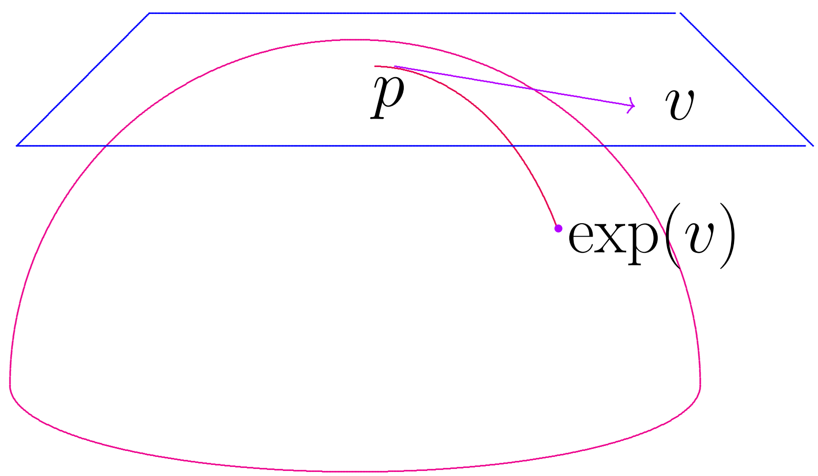

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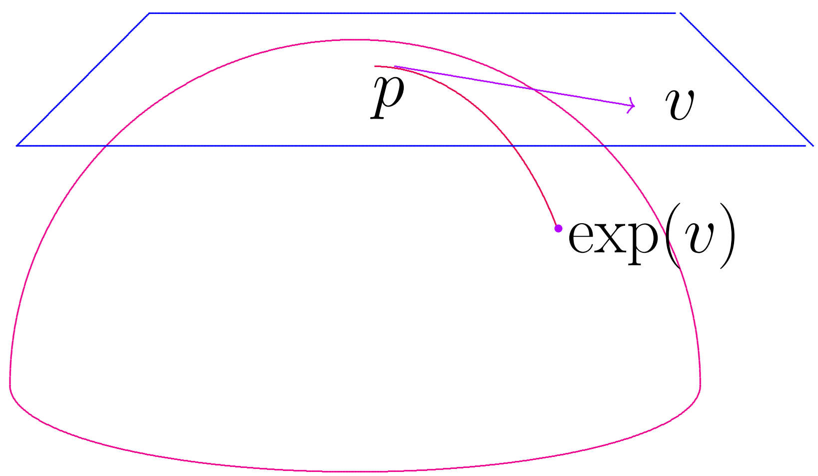

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For G with bi-invariant g, equals Lie-theoretic exp.

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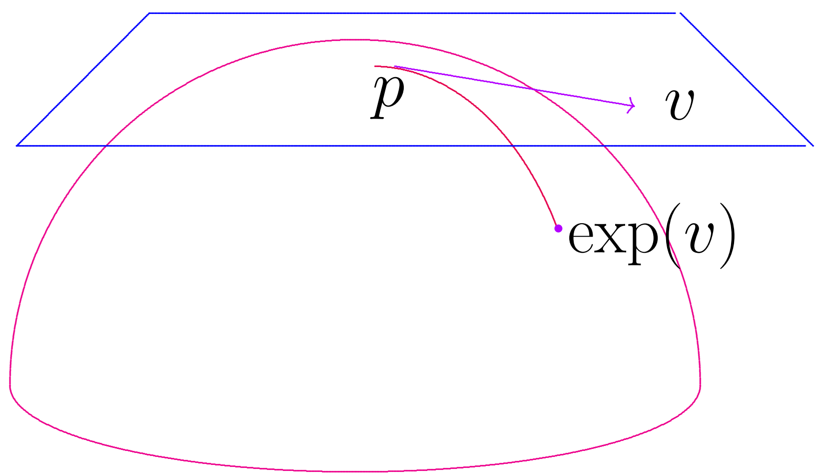

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