MAT 552

Introduction to

Lie Groups and Lie Algebras

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$$v \neq 0 \quad \Longrightarrow \quad g(v, v) > 0.$$

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We say that γ is a path from p to q if

$$\gamma(a) = p \quad \text{and} \quad \gamma(b) = q.$$

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$$(\nabla g)(u, v, w) := u g(v, w) - g(\nabla_u v, w) - g(v, \nabla_u w)$$

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Proof also works if g is just pseudo-Riemannian:

$$v \neq 0 \implies \exists w \text{ } s.t. \text{ } g(v, w) \neq 0.$$

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Of course, such curves can more generally be defined on any manifold with affine connection ∇.

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These are the "one-parameter subgroups" of G.

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Same conclusion: geodesics through e are the

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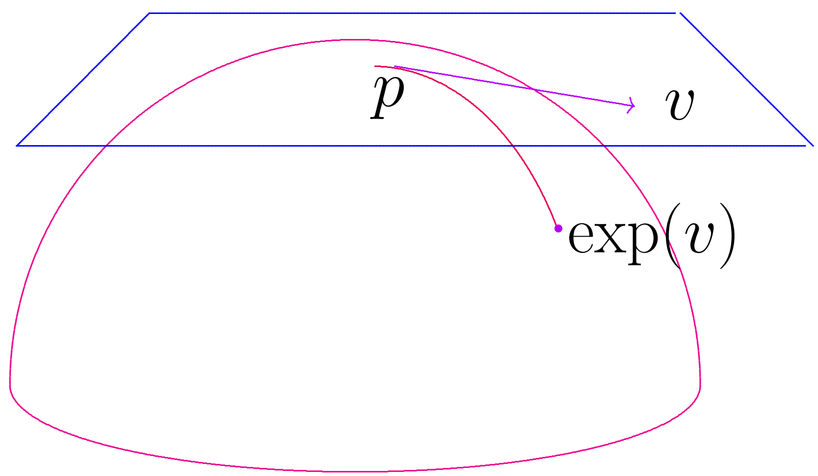

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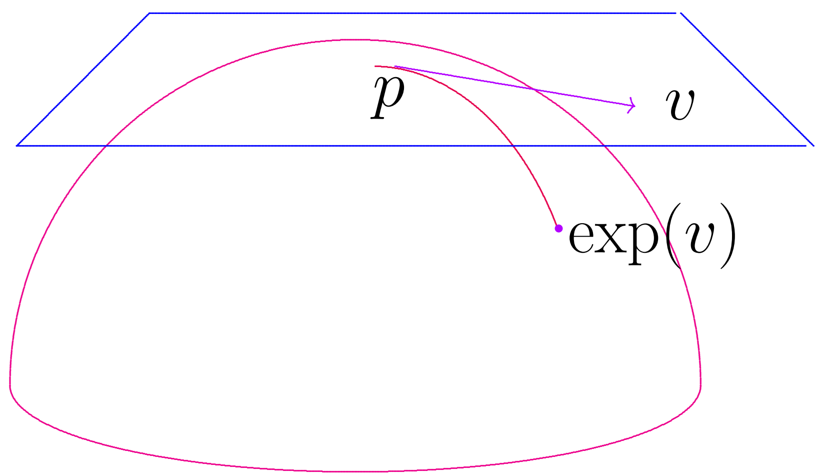

This is a diffeomorphism on a neighborhhood of 0.

For G with bi-invariant g, equals Lie-theoretic exp.

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Any of these implies

- *any* $p, q \in M$ *joined by a minimizing geodesic segment.*