MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

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and we get a representation $Ad_* : \mathsf{G} \to \mathbf{GL}(T_{\mathsf{e}}\mathsf{G})$ called the adjoint representation.

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To find out, we'll use the **Lie derivative**.

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When V not complete, "flow" only defined on a neighborhood of $M \times \{0\} \subset M \times \mathbb{R}$:

$$\Phi : M \times \mathbb{R} \dashrightarrow M$$

where dashed arrow means "not defined everywhere."

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Lie derivative of tensor field φ w/resp. to V:

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Lie bracket:

$$[V, W]f = V(Wf) - W(Vf).$$

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