MAT 552

Introduction to

Lie Groups and Lie Algebras

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March 18, 2021

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$$v \neq 0 \quad \Longrightarrow \quad g(v, v) > 0.$$

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We say that γ is a path from p to q if

$$\gamma(a) = p \quad \text{and} \quad \gamma(b) = q.$$

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$$(\nabla g)(u, v, w) := ug(v, w) - g(\nabla_u v, w) - g(v, \nabla_u w)$$

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Proof also works if g is just pseudo-Riemannian:

$$v \neq 0 \implies \exists w \text{ s.t. } g(v, w) \neq 0.$$

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Of course, such curves can more generally be defined on any manifold with affine connection ∇.

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which is at least defined in a neighborhood of 0: