MAT 552

Introduction to

Lie Groups and Lie Algebras

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Proof. Partition of unity argument. . .

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$$(\nabla g)(u, v, w) := u g(v, w) - g(\nabla_u v, w) - g(v, \nabla_u w)$$

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