MAT 552

Introduction to

Lie Groups and Lie Algebras

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$$v \neq 0 \quad \Longrightarrow \quad g(v, v) > 0.$$

Example. Since $T_p\mathbb{R}^n = \mathbb{R}^n$, usual dot product $\langle\ ,\ \rangle$ defines a Riemannian metric on $\mathbb{R}^n$:

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Restricting dot product $\langle\ ,\ \rangle$ to TM defines a Riemannian metric on M:

$$g = j^*(dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n).$$

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May then define an affine connection ∇ on M by

$$\nabla_u v = (D_u v)^{\|}$$

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$$\mathcal{T}_{\nabla}(v, w) := \nabla_v w - \nabla_w v - [v, w]$$

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$$(\nabla g)(u, v, w) := ug(v, w) - g(\nabla_u v, w) - g(v, \nabla_u w)$$

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$$ug(v, w) + vg(w, u) - wg(u, v) =$$

$$g(\nabla_u v, w) + g(v, \nabla_u w) + g(\nabla_v w, u)$$

$$+g(w, \nabla_v u) - g(\nabla_w u, v) - g(u, \nabla_w v)$$

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$$2g(w, \nabla_u v) = ug(v,w) + vg(w,u) - wg(u,v) \\ +g(v,[w,u]) + g(w,[u,v]) - g(u,[v,w])$$

Proof. Torsion-free: $\nabla_v w - \nabla_w v = [v, w]$

$$g(w, \nabla_u v) = \frac{1}{2}\Big[ug(v, w) + vg(w, u) - wg(u, v) \\ + g(v, [w, u]) + g(w, [u, v]) - g(u, [v, w])\Big]$$