Homework # 2

MAT 552

Due 4/21/21 at 11 pm

Do at least five problems. Extra credit for doing more!

1. Let (M, g) and (N, h) be connected Riemannian manifolds of the same dimension n, and suppose that

$$\Phi : M \to N$$

is a smooth map such that $\Phi^* h = g$. (Such a map is called a *local isometry.*)

(a) Prove that Φ is a local diffeomorphism.

(b) If γ is a geodesic in (M, g), show that $\Phi \circ \gamma$ is a geodesic in (N, h).

(c) If (M, g) is geodesically complete at p, show that (N, h) is geodesically complete at $\Phi(p)$.

(d) If (M, g) is geodesically complete, use the Hopf-Rinow Theorem to show that Φ must therefore be surjective.

(e) If (M, g) is complete, show that Φ must be a covering map. (To do this, first show that the inverse image of a geodesically convex ball $B_\varepsilon(q)$ of sufficiently small radius ε about $q \in N$ is the union of the balls $B_\varepsilon(p_j)$, where $\Phi^{-1}(\{q\}) = \{p_j\} \subset M$. Then show that these balls in M are actually disjoint, and that each is carried diffeomorphically to $B_\varepsilon(q)$ by Φ.)

2. Let (M, g) be a connected Riemannian n-manifold, and let

$$\Phi : M \to M$$

be a smooth map such that $\Phi^* g = g$. Such a metric-compatible map will be called a *local self-isometry* of the Riemannian manifold (M, g). If Φ is also a diffeomorphism, one then says that Φ is an *isometry* of (M, g).

If (M, g) is complete and simply connected, use problem 1 to show that any local self-isometry Φ of (M, g) is a diffeomorphism. Then show that this implies that Φ and Φ^{-1} are both isometries, and that Φ preserves Riemannian distance, in the sense that, for any $p, q \in M$,

$$\mathsf{dist}(\Phi(p), \Phi(q)) = \mathsf{dist}(p, q). \tag{1}$$

3. Let (N, h) be a complete connected Riemannian manifold, and let

$$\varpi : M \to N$$

be a covering map. Equip M with the pulled-back metric $g = \varpi^* h$, so that ϖ becomes a local isometry in the sense of problem 1. Use the Hopf-Rinow Theorem to show that (M, g) is then a complete Riemannian manifold.

4. Let (M, g) be a connected Riemannian manifold. If $\Phi : M \to M$ is a set-theoretic function that preserves Riemannian distance in the sense of (1), show that Φ is smooth, and is an isometry in the sense of problem 2.

5. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be its Killing form, as defined by

$$B(X, Y) = \mathsf{trace}\ (Ad_X \circ Ad_Y). \tag{2}$$

Show that

$$B([X, Y], Z) + B(Y, [X, Z]) = 0$$

for any $X, Y, Z \in \mathfrak{g}$, and then use this to prove that B defines a bi-invariant symmetric tensor field on G. If G is compact and the center $\mathfrak{z} \subset \mathfrak{g}$ is zero, then give a Riemannian explanation for the bi-invariance of B, and then use this approach to show that $-B$ is a bi-invariant Riemannian metric on G.

6. The Lie group $\mathbf{SU}(n)$ is compact, and so admits a bi-invariant Riemannian metric g. Show that while such a metric always has strictly positive Ricci curvature, it does *not* have (strictly) positive sectional curvature if $n \geq 3$.

7. Recall that a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is by definition a $\mathbb{C}$-vector space $\mathfrak{g}$, equipped with a skew-symmetric complex-bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that satisfies the Jacobi identity. Since a complex-bilinear map is also real-bilinear, this means that $\mathfrak{g}$ may also be viewed as a Lie algebra $\mathfrak{g}_{\mathbb{R}}$ over $\mathbb{R}$.

In view of this, (2) actually defines *two* versions, $B_{\mathbb{R}}$ and $B_{\mathbb{C}}$, of the Killing form, depending on whether the relevant linear algebra is carried out over $\mathbb{R}$ or $\mathbb{C}$. How are these related? State and prove a simple formula that expresses $B_{\mathbb{R}}$ in terms of $B_{\mathbb{C}}$.

8. If $\mathbb{V}$ is a real vector space, the complex vector space $\mathbb{V} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{V} \oplus i\mathbb{V}$ is called the *complexification* of $\mathbb{V}$. Given a real Lie algebra $\mathfrak{g}$, prove that the vector space $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ can be made into a complex Lie algebra in a unique way such that restricting the Lie bracket from $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{g} = \mathfrak{g} \oplus i\mathsf{0}$ is the original Lie bracket on $\mathfrak{g}$. The resulting complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is called the *complexification* of $\mathfrak{g}$, while $\mathfrak{g}$ is called a *real form* of $\mathfrak{g}_{\mathbb{C}}$.

Prove that $\mathfrak{su}(n)$ and $\mathfrak{sl}(n, \mathbb{R})$ are both real forms of $\mathfrak{sl}(n, \mathbb{C})$.

9. Let G be a real Lie group, let $\mathsf{G}_{\mathbb{C}}$ be a complex Lie group, and suppose that we are given a smooth embedding $\mathsf{G} \hookrightarrow \mathsf{G}_{\mathbb{C}}$ as a closed Lie subgroup. We then say that G is a *real form* of $\mathsf{G}_{\mathbb{C}}$, and that $\mathsf{G}_{\mathbb{C}}$ is a *complexification* of G, if the Lie algebra of $\mathsf{G}_{\mathbb{C}}$ is isomorphic to the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, in a manner that identifies $\mathfrak{g} \oplus i\mathsf{0}$ with image of $\mathfrak{g}$ under the derivative of the embedding.

Show that $\mathbf{SL}(n, \mathbb{R})$ and $\mathbf{SU}(n)$ are both real forms of $\mathbf{SL}(n, \mathbb{C})$. How are the Killing forms of these three Lie groups related?

10. According to the definition used in class,

$$\mathbf{Sp}(n) = \mathbf{GL}(n, \mathbb{H}) \cap \mathbf{O}(4n)$$

where $\mathbb{H} = \mathbb{R}^4$ denotes the non-commutative field of quaternions, and where $\mathbf{GL}(n, \mathbb{H})$ is the group of real-linear transformations L of $\mathbb{H}^n = \mathbb{R}^{4n}$ which are *right-quaternionic-linear*, in the sense that

$$L(\vec{v}q) = L(\vec{v})q \quad \forall \vec{v} \in \mathbb{H}^n,\ q \in \mathbb{H}.$$

(a) Prove that the above definition is equivalent to the following:

$$\mathbf{Sp}(n) = \{L \in \mathbf{U}(2n) \mid L^*\Omega = \Omega\}$$

where Ω is the complex-valued 2-form

$$\Omega = dz^1 \wedge dz^2 + dz^3 \wedge dz^4 + \cdots + dz^{2n-1} \wedge dz^{2n}$$

on $\mathbb{C}^{2n}$, where $(z^1, \ldots, z^{2n})$ are the standard complex coordinates.

(b) Let

$$\mathbf{Sp}(n, \mathbb{C}) = \{L \in \mathbf{GL}(2n, \mathbb{C}) \mid L^*\Omega = \Omega\},$$

where Ω is the complex-valued 2-form defined in part (a). Prove that $\mathbf{Sp}(n, \mathbb{C})$ is a complex Lie group, and is a complexification of $\mathbf{Sp}(n)$.

(c) Let

$$\mathbf{Sp}(n, \mathbb{R}) = \{L \in \mathbf{GL}(2n, \mathbb{R}) \mid L^*\omega = \omega\},$$

where ω is the real-valued 2-form on $\mathbb{R}^{2n}$ given by

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \cdots + dx^{2n-1} \wedge dx^{2n}. \tag{3}$$

Prove that $\mathbf{Sp}(n, \mathbb{R})$ is yet another real form of $\mathbf{Sp}(n, \mathbb{C})$.

Historical note: Closed non-degenerate 2-forms modeled on (3) are called *symplectic forms*, and the abbreviation **Sp** stands for symplectic. This term was coined by Hermann Weyl as a sophisticated parody of complex:

com (Latin) + plex (Greek) = together + braided = *braided together*

sym (Greek) + plectic (Latin) = together + braided = *braided together*