MAT 552

Introduction to

Lie Groups and Lie Algebras

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I'll typically denote identity element by $\mathsf{e} \in \mathsf{G}$.

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We previously also saw not simply connected.

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Transverse to zero, so manifold of dimension n^2.

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So this Lie group has dimension n^2.

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Notice that $\mathbf{U}(n)$ is compact!

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Consider loop

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$$\mathbf{SU}(2) = \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \middle| \, |z_1|^2 + |z_2|^2 = 1 \right\}$$

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We'll later show: $\mathbf{SU}(n)$ is simply connected $\forall n$.

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Transverse to zero, so manifold of dimension $\frac{n(n-1)}{2}$.

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Notice that $\mathbf{O}(n)$ is compact!

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$(\det \mathbf{A})^2 = (\det \mathbf{A}^t)(\det \mathbf{A}) = \det(\mathbf{A}^t\mathbf{A}) = \det \mathbf{I} = 1$

so

$$\det \mathbf{A} \in \{\pm 1\}.$$

Both possibilities occur, because

$$\begin{pmatrix} \pm 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad .$$

Example. Let $\mathsf{G} = \mathbf{O}(n)$ be orthogonal group:

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$$\mathbf{SO}(n) = \{\mathbf{A} \in \mathbf{O}(n) \quad | \quad \det \mathbf{A} = 1\}$$

Proposition. $\mathbf{SO}(n)$ *is connected.*

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Lemma. *Let* $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ *be any orthogonal transformation. Then there is an orthonormal basis for* $\mathbb{R}^n$ *in which* $\mathbf{A}$ *is represented by a matrix of the form*

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & & & & \\ \sin\theta_1 & \cos\theta_1 & & & & \\ & & \cos\theta_2 & -\sin\theta_2 & & \\ & & \sin\theta_2 & \cos\theta_2 & & \\ & & & & \ddots & \\ & & & & & (\pm 1) \end{bmatrix}$$

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For $\mathbf{A} \in \mathbf{SO}(n)$, connect to $\mathbf{I}$ by

$$\mathbf{A}(t) = \begin{bmatrix} \cos t\theta_1 & -\sin t\theta_1 & & & & \\ \sin t\theta_1 & \cos t\theta_1 & & & & \\ & & \cos t\theta_2 & -\sin t\theta_2 & & \\ & & \sin t\theta_2 & \cos t\theta_2 & & \\ & & & & \ddots & \\ & & & & & (+1) \end{bmatrix}$$

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More generally, we'll prove

$$\pi_1(\mathbf{SO}(n)) \cong \mathbb{Z}_2.$$

Proposition. *As smooth manifolds,*

$$\mathbf{GL}(n,\mathbb{R}) \approx \mathbf{O}(n) \times \mathbb{R}^{n(n+1)/2}$$

$$\mathbf{SL}(n,\mathbb{R}) \approx \mathbf{SO}(n) \times \mathbb{R}^{(n^2+n-2)/2}$$

$$\mathbf{GL}(n,\mathbb{C}) \approx \mathbf{U}(n) \times \mathbb{R}^{n^2}$$

$$\mathbf{SL}(n,\mathbb{C}) \approx \mathbf{SU}(n) \times \mathbb{R}^{n^2-1}$$