MAT 552

Introduction to

Lie Groups and Lie Algebras

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$$e_1 = \mathbf{1}, \quad e_2 = \mathbf{i}, \quad e_3 = \mathbf{j}, \quad e_4 = \mathbf{k}$$

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Quaternionic conjugation

$$\overline{t\,\mathbf{1} + u\,\mathbf{i} + v\,\mathbf{j}, + w\,\mathbf{k}} = t\,\mathbf{1} - u\,\mathbf{i} - v\,\mathbf{j}, - w\,\mathbf{k}$$

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Notice that

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So any $q \neq 0$ has multiplicative inverse

$$q^{-1} = \frac{1}{\|q\|^2}\overline{q}.$$

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Also notice that conjugation satisfies

$$\overline{q_1 q_2} = \overline{q_2}\ \overline{q_1}$$

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Fundamental Groups of "Classical" Compact Lie Groups

- $\mathbf{SO}(2) = \mathbf{U}(1) = S^1$
  $\pi_1(\mathbf{SO}(2)) = \pi_1(\mathbf{U}(1)) = \mathbb{Z}$

- $\mathbf{SO}(3) \approx \mathbb{RP}^3$
  $\pi_1(\mathbf{SO}(3)) \cong \mathbb{Z}_2$

  $\mathbf{SO}(3) \hookrightarrow \mathbf{SO}(n),\ n \geq 3$
  We'll show this induces:
  $\pi_1(\mathbf{SO}(n)) \cong \mathbb{Z}_2,\ n \geq 3$

Fiber bundle:

$$\begin{array}{c} \mathbf{SO}(n+1) \\ \downarrow \\ S^n \end{array}$$

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Homotopy exact sequence of a fibration

$$\cdots \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \cdots$$

$\pi_2(X)$ = homotopy classes of maps $S^2 \to X$

$\pi_0(X)$ = set of connected components of X

Need base point to define relevant structures!

Example of a "Serre fibration":

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

Homotopy exact sequence of a fibration

$$\cdots \to \pi_{k+1}(B) \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots$$

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Example of a "Serre fibration":

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For $n \geq 3$, get isomorphism

$$\pi_1(\mathbf{SO}(n)) \to \pi_1(\mathbf{SO}(n+1))$$

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By induction,

$$\pi_1(\mathbf{SO}(n)) \cong \pi_1(\mathbf{SO}(3)) \cong \mathbb{Z}_2, \ \forall n \geq 3$$

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- $\pi_1(\mathbf{SO}(n)) \cong \mathbb{Z}_2,\ n \geq 3$
- $\mathbf{SU}(2) \approx S^3$
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