MAT 552

Introduction to

Lie Groups and Lie Algebras

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Stony Brook University

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So any $q \neq 0$ has multiplicative inverse

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Also notice that conjugation satisfies

$$\overline{q_1 q_2} = \overline{q_2}\ \overline{q_1}$$

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