MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
Stony Brook University

February 4, 2021

Definition. *A Lie group* G *is a smooth manifold that is also equipped with a group structure,*

Definition. *A* *Lie group* G *is a smooth manifold that is also equipped with a group structure, in such a way that these two structures are compatible, in the precise sense that*

Definition. *A Lie group* G *is a smooth manifold that is also equipped with a group structure, in such a way that these two structures are compatible, in the precise sense that*

- *The group multiplication operation*

$$\begin{aligned} \mathsf{G} \times \mathsf{G} &\longrightarrow \mathsf{G} \\ (\mathsf{a},\ \mathsf{b}) &\mapsto \mathsf{ab} \end{aligned}$$

  is a smooth map; and

Definition. *A* *Lie group* G *is a smooth manifold that is also equipped with a group structure, in such a way that these two structures are compatible, in the precise sense that*

- *The group multiplication operation*

$$\begin{aligned} \mathsf{G} \times \mathsf{G} &\longrightarrow \mathsf{G} \\ (\mathsf{a},\ \mathsf{b}) &\mapsto \mathsf{ab} \end{aligned}$$

  is a smooth map; and

- *the group inversion operation*

$$\begin{aligned} \mathsf{G} &\longrightarrow \mathsf{G} \\ \mathsf{a} &\mapsto \mathsf{a}^{-1} \end{aligned}$$

  is also a smooth map.

Definition. *A Lie group* G *is a smooth manifold that is also equipped with a group structure, in such a way that these two structures are compatible, in the precise sense that*

- *The group multiplication operation*

$$\begin{aligned} \mathsf{G} \times \mathsf{G} &\longrightarrow \mathsf{G} \\ (\mathsf{a}, \mathsf{b}) &\mapsto \mathsf{ab} \end{aligned}$$

  is a smooth map; and

- *the group inversion operation*

$$\begin{aligned} \mathsf{G} &\longrightarrow \mathsf{G} \\ \mathsf{a} &\mapsto \mathsf{a}^{-1} \end{aligned}$$

  is also a smooth map.

I'll typically denote identity element by $\mathsf{e} \in \mathsf{G}$.

Example. Recall that n-dimensional Euclidean space $\mathbb{R}^n$ is the proto-typical example of a smooth n-manifold.

it also forms a group with respect to vector addition

$$(x^1, \ldots, x^n) + (y^1, \ldots, y^n) = (x^1 + y^1, \ldots, x^n + y^n)$$

for which the inversion operation is

$$(x^1, \ldots, x^n) \mapsto -(x^1, \ldots, x^n) = (-x^1, \ldots, -x^n).$$

Since these operations are smooth, they make

$$\mathsf{G} = \mathbb{R}^n$$

into a Lie group.

By the same reasoning, any finite-dimensional real vector space is a Lie group.

Note, however, that these examples are all Abelian!

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{R})$ be the set of invertible $n \times n$ **real** matrices:

$$\mathbf{GL}(n, \mathbb{R}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \middle| \det \mathbf{A} \neq 0 \right\}$$

Manifold because open set in $\mathbb{R}^{n^2}$.

Group under matrix multiplication.

This example is non-compact.

Also, this example is not connected.

Determinant can be $+$ or $-$.

Will later show $\exists$ exactly two components...

Theorem. *Let* G *be any Lie group, and let* $\mathsf{G}_0 \subset \mathsf{G}$ *be the connected component containing the identity element* e*. Then* G_0 *is a normal subgroup of* G*. In particular,* G_0 *is itself a Lie group.*

Example. Let $G = \mathbf{GL}(n, \mathbb{R})$ be the set of invertible $n \times n$ **real** matrices:

$$\mathbf{GL}(n,\mathbb{R}) = \left\{ \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

Manifold because open set in $\mathbb{R}^{n^2}$.

Group under matrix multiplication.

This example is non-compact.

Also, this example is not connected.

Determinant can be $+$ or $-$.

Will later show $\exists$ exactly two components...

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{R})$ be the set of invertible $n \times n$ **real** matrices:

$$\mathbf{GL}(n, \mathbb{R}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \middle| \quad \det \mathbf{A} \neq 0 \right\}$$

Two substantially different, but equally important, pictures to keep in mind:

$$\begin{aligned} \mathbf{GL}(n, \mathbb{R}) &= \{\text{vector-space isomorphisms } \mathbb{R}^n \to \mathbb{R}^n\} \\ &= \{\text{bases for } \mathbb{R}^n\} \end{aligned}$$

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{C})$ be the set of invertible $n \times n$ complex matrices:

$$\mathbf{GL}(n, \mathbb{C}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

Manifold because open set in $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$.

In fact, complex manifold because open set in $\mathbb{C}^{n^2}$.

Group under matrix multiplication.

So Lie group.

Also a complex Lie group.

Definition. *A complex Lie group* G *is a complex manifold that is also equipped with a group structure, in such a way that these two structures are compatible, in the precise sense that*

- *The group multiplication operation*

$$\begin{aligned} \mathsf{G} \times \mathsf{G} &\longrightarrow \mathsf{G} \\ (\mathsf{a},\ \mathsf{b}) &\mapsto \mathsf{ab} \end{aligned}$$

  is a holomorphic map; and

- *the group inversion operation*

$$\begin{aligned} \mathsf{G} &\longrightarrow \mathsf{G} \\ \mathsf{a} &\mapsto \mathsf{a}^{-1} \end{aligned}$$

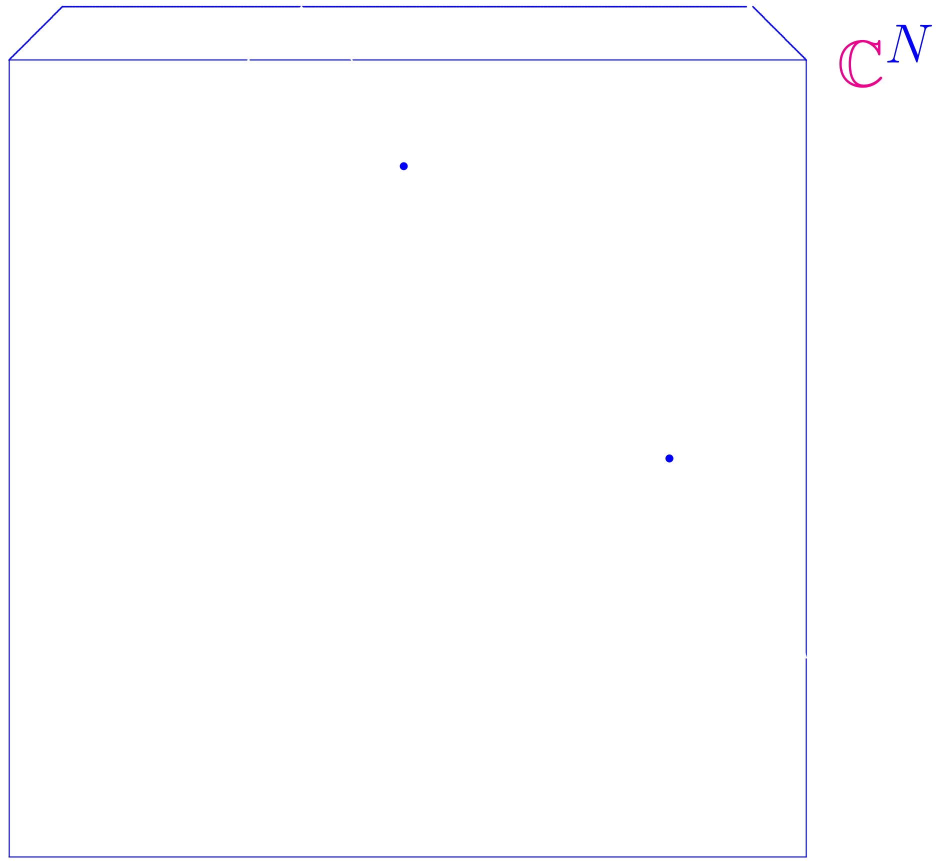
  is also a holomorphic map.

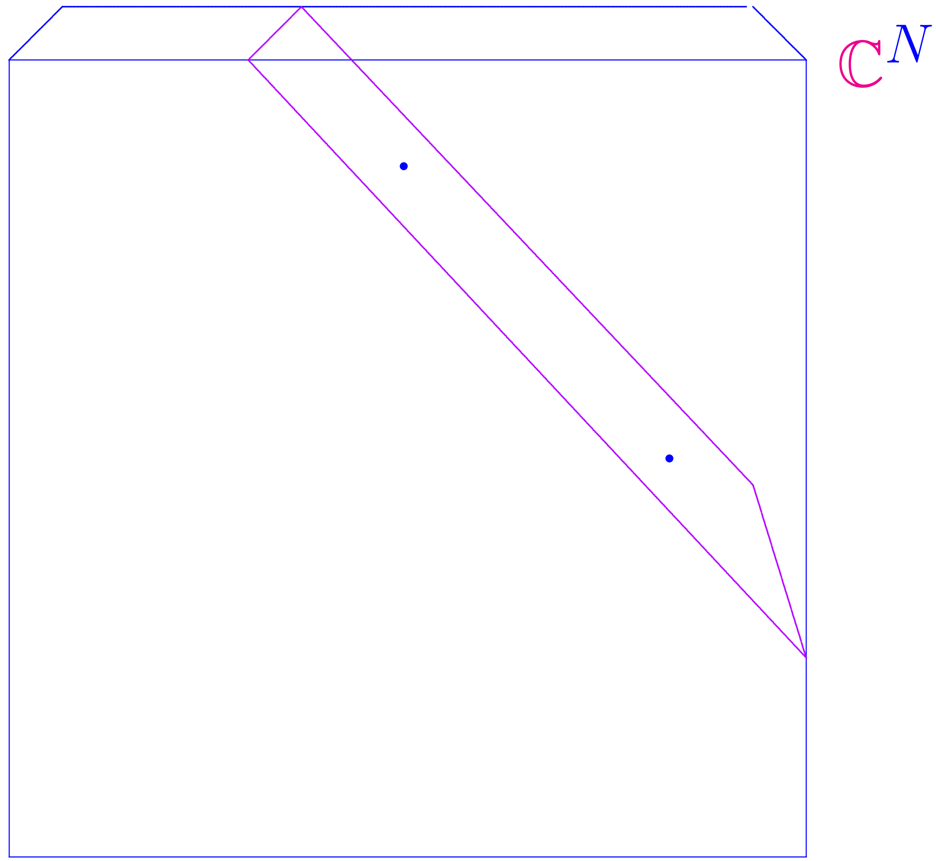
Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{C})$ be the set of invertible $n \times n$ complex matrices:

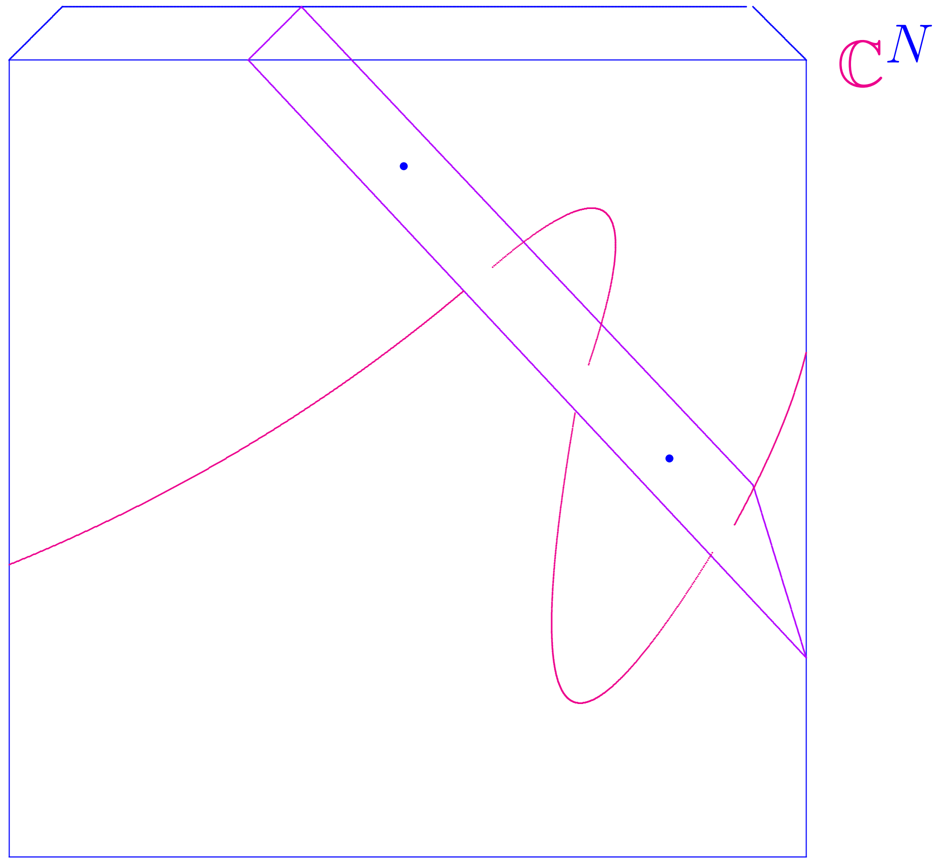
$$\mathbf{GL}(n, \mathbb{C}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

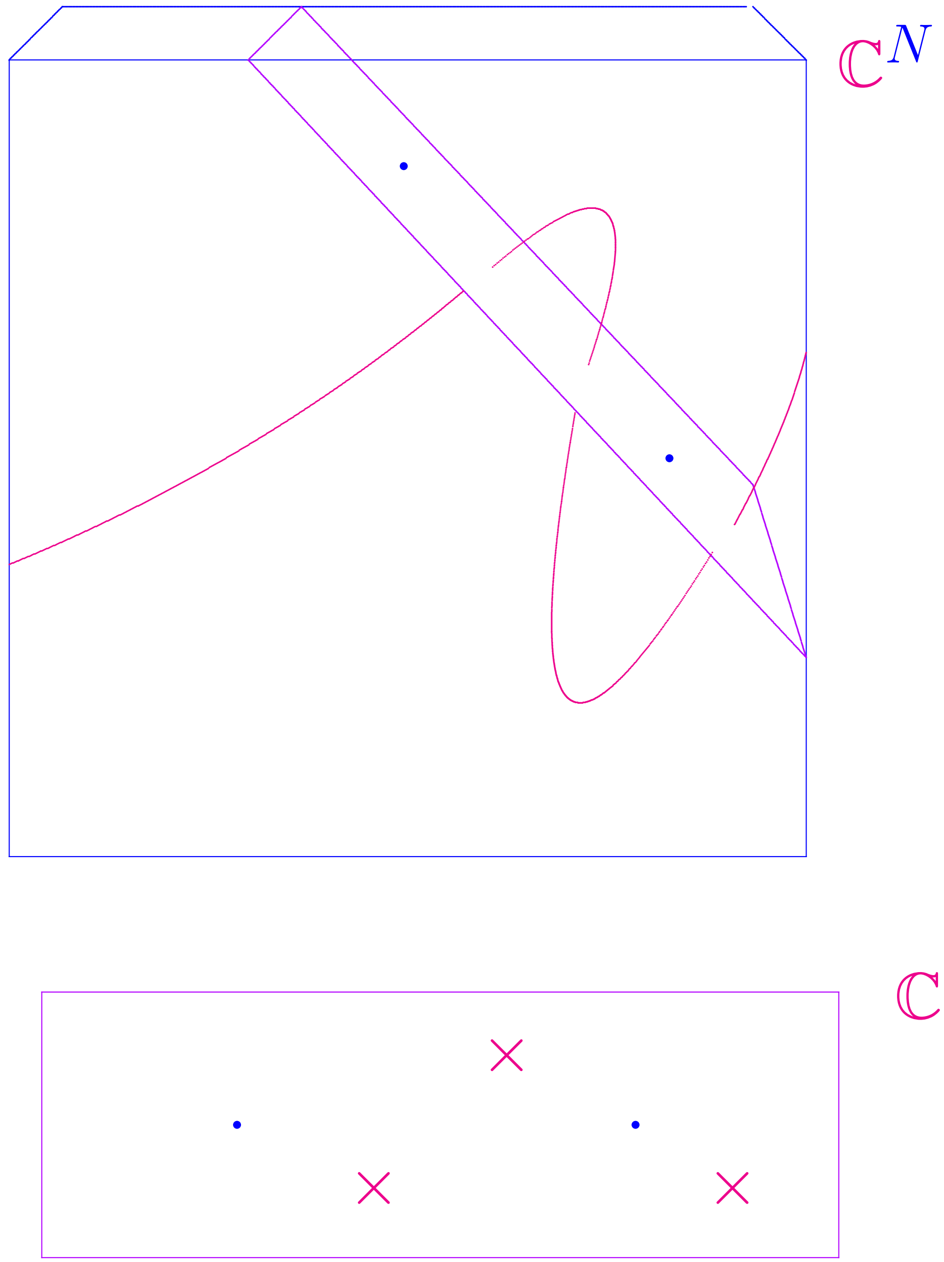
Complement of zeroes of polynomial $\mathbb{C}^N \to \mathbb{C}$.

So connected!

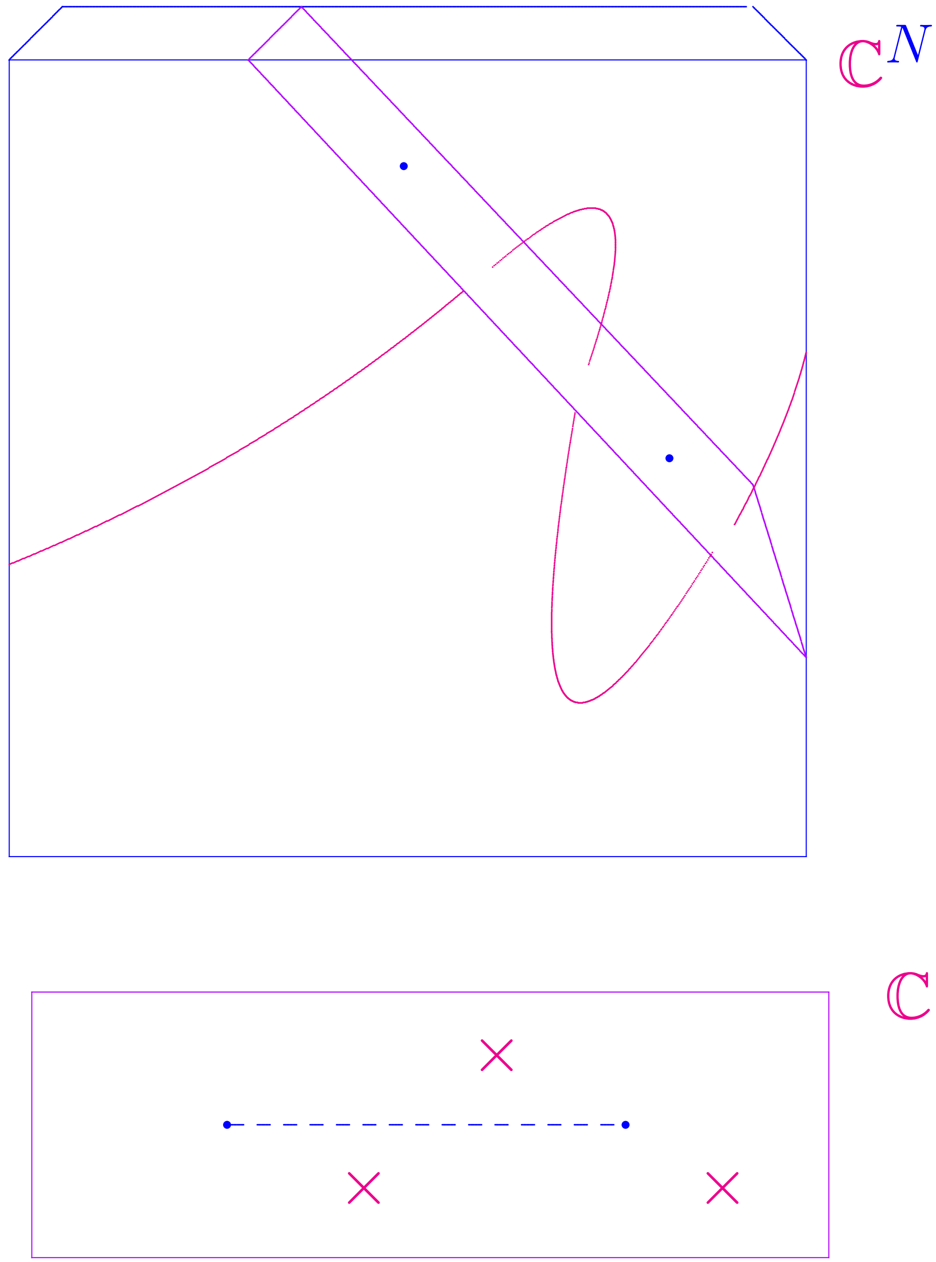

$\mathbb{C}^N$


$\mathbb{C}^N$


$\mathbb{C}^N$



$\mathbb{C}^N$
$\mathbb{C}$



$\mathbb{C}^N$
$\mathbb{C}$

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{C})$ be the set of invertible $n \times n$ complex matrices:

$$\mathbf{GL}(n, \mathbb{C}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

Complement of zeroes of polynomial $\mathbb{C}^N \to \mathbb{C}$.

So connected!

But not simply-connected!

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{C})$ be the set of invertible $n \times n$ complex matrices:

$$\mathbf{GL}(n, \mathbb{C}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

Complement of zeroes of polynomial $\mathbb{C}^N \to \mathbb{C}$.

So connected!

But not simply-connected!

$$\det \begin{bmatrix} e^{i\theta} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = e^{i\theta}$$

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{C})$ be the set of invertible $n \times n$ complex matrices:

$$\mathbf{GL}(n, \mathbb{C}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

Complement of zeroes of polynomial $\mathbb{C}^N \to \mathbb{C}$.

So connected!

But not simply-connected!

$$\det : \mathbf{GL}(n, \mathbb{C}) \to \mathbb{C} - \{0\}$$

induces surjection $\pi_1(\mathbf{GL}(n, \mathbb{C})) \to \mathbb{Z}$.

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{C})$ be the set of invertible $n \times n$ complex matrices:

$$\mathbf{GL}(n, \mathbb{C}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

Two substantially different, but equally important, pictures to keep in mind:

$$\mathbf{GL}(n, \mathbb{C}) = \{\text{vector-space isomorphisms } \mathbb{C}^n \to \mathbb{C}^n\}$$

Example. Let $\mathsf{G} = \mathbf{GL}(n, \mathbb{C})$ be the set of invertible $n \times n$ complex matrices:

$$\mathbf{GL}(n,\mathbb{C}) = \left\{ \mathbf{A} = \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} \;\middle|\; \det \mathbf{A} \neq 0 \right\}$$

Two substantially different, but equally important, pictures to keep in mind:

$$\begin{aligned} \mathbf{GL}(n,\mathbb{C}) &= \{\text{vector-space isomorphisms } \mathbb{C}^n \to \mathbb{C}^n\} \\ &= \{\text{bases for } \mathbb{C}^n\} \end{aligned}$$

Example. Let $\mathsf{G} = \mathbf{U}(n)$ be unitary group:

$$\begin{aligned}
\mathbf{U}(n) &= \{\text{length-preserving } \mathbb{C}\text{-vector-space isom'isms } \mathbb{C}^n \to \mathbb{C}^n\} \\
&= \{\langle\ ,\ \rangle\text{-preserving vector-space isomorphisms } \mathbb{C}^n \to \mathbb{C}^n\} \\
&= \{\text{orthonormal bases for } \mathbb{C}^n\} \\
&= \{\mathbf{A} \in \mathbf{GL}(n, \mathbb{C}) \quad | \quad \mathbf{A}^*\mathbf{A} = \mathbf{I}\}
\end{aligned}$$

$$\begin{bmatrix} \bar{\mathsf{a}}_{11} & \cdots & \bar{\mathsf{a}}_{n1} \\ \vdots & & \vdots \\ \bar{\mathsf{a}}_{1n} & \cdots & \bar{\mathsf{a}}_{nn} \end{bmatrix} \begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & & \vdots \\ \mathsf{a}_{n1} & \cdots & \mathsf{a}_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$\mathbf{A}^*\mathbf{A}$ automatically Hermitian.

So cut out by n^2 real equations in $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$.

Transverse to zero, so manifold of $\mathbb{R}$-dimension n^2.