# *MAT 552*

# *Introduction to*

# *Lie Groups and Lie Algebras*

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This map is called the *flow* of $V$.

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I'll typically denote identity element by $\mathsf{e} \in \mathsf{G}$.

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So as $\mathsf{a} \in \mathsf{G}$ varies, we get an isomorphism

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Thus, as a vector space

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Thus, we have now shown that every Lie group $\mathsf{G}$ has an associated Lie algebra $\mathfrak{g}$, with underlying vector space $T_{\mathsf{e}}\mathsf{G}$.

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**Proposition.** *The Lie algebra of* $\mathsf{G} = \mathbf{GL}(n, \mathbb{R})$ *is naturally isomorphic to*

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For any $\mathsf{A} \in \mathfrak{gl}(n, \mathbb{R})$, corresponding one-parameter subgroup is

$$\gamma(t) = e^{t\mathsf{A}}.$$