MAT 552

Introduction to

Lie Groups and Lie Algebras

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I'll typically denote identity element by $\mathsf{e} \in \mathsf{G}$.

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So as $\mathsf{a} \in \mathsf{G}$ varies, we get an isomorphism

$$\mathsf{G} \times T_{\mathsf{e}}\mathsf{G} \longrightarrow T\mathsf{G}.$$

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Thus, as a vector space

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$$[X + Y, Z] = [X, Z] + [Y, Z], \quad [tX, Y] = t[X, Y], \quad \text{etc.}$$

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Thus, we have now shown that every Lie group G has an associated Lie algebra $\mathfrak{g}$, with underlying vector space $T_{\mathsf{e}}\mathsf{G}$.

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Definition. *The Lie algebra* $\mathfrak{g}$ *of the Lie group* G *is by definition*

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Proposition. *The Lie algebra of* $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ *is naturally isomorphic to*

$$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \{n \times n \textit{ real matrices } \mathsf{A}\}$$

equipped with the operation $[\ ,\ ]$ *defined by*

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because $\mathbf{GL}(n, \mathbb{R})$ is an open set in the $n \times n$ real matrices. But there is still something to check, because this doesn't tell us what the bracket operation is!

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Coordinatize $\mathbf{GL}(n, \mathbb{R})$ by

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Left-invariant vector fields:

$$Y_j^k \mathsf{A}_i^j \frac{\partial}{\partial Y_i^k}$$

with value at the matrix Y given by matrix $Y\mathsf{A}$.

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$$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \{n \times n \textit{ real matrices } \mathsf{A}\}$$

equipped with the operation $[\ ,\ ]$ *defined by*

$$[\mathsf{A}, \mathsf{B}] = \mathsf{A}\mathsf{B} - \mathsf{B}\mathsf{A}.$$

$$\left[Y^k_j \mathsf{A}^j_i \frac{\partial}{\partial Y^k_i}, Y^c_b \mathsf{B}^b_a \frac{\partial}{\partial Y^c_a} \right] = Y^k_j \mathsf{A}^j_i (\delta^c_k \delta^i_b) \mathsf{B}^b_a \frac{\partial}{\partial Y^c_a} - Y^c_b \mathsf{B}^b_a (\delta^k_c \delta^a_j) \mathsf{A}^j_i \frac{\partial}{\partial Y^k_i}$$

$$= Y^k_j \mathsf{A}^j_b \mathsf{B}^b_a \frac{\partial}{\partial Y^k_a} - Y^k_b \mathsf{B}^b_a \mathsf{A}^a_i \frac{\partial}{\partial Y^k_i}$$

$$= Y^k_j \mathsf{A}^j_\ell \mathsf{B}^\ell_i \frac{\partial}{\partial Y^k_i} - Y^k_j \mathsf{B}^j_\ell \mathsf{A}^\ell_i \frac{\partial}{\partial Y^k_i}$$

$$= Y^k_j \, [\mathsf{A}, \mathsf{B}]^j_i \, \frac{\partial}{\partial Y^k_i}$$

QED

Proposition. *The Lie algebra of* $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ *is naturally isomorphic to*

$$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \{n \times n \textit{ real matrices } \mathsf{A}\}$$

equipped with the operation $[\ ,\ ]$ *defined by*

$$[\mathsf{A}, \mathsf{B}] = \mathsf{AB} - \mathsf{BA}.$$

This now also gives us the right to also represent Lie algebras of Lie subgroups $\mathsf{G} \subset \mathbf{GL}(n, \mathbb{R})$ as Lie algebras of matrices...