MAT 552

Introduction to

Lie Groups and Lie Algebras

Claude LeBrun
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February 11, 2021

Announcement:

Next Tuesday, the class will start at rough 10 am because I will be giving a seminar that morning.

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I'll typically denote identity element by $\mathsf{e} \in \mathsf{G}$.

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In fact, the only spheres with trivial tangent bundle are

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So as $\mathsf{a} \in \mathsf{G}$ varies, we get an isomorphism

$$\mathsf{G} \times T_{\mathsf{e}}\mathsf{G} \longrightarrow T\mathsf{G}.$$

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Thus, as a vector space

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Thus, if X and Y are vector fields on G which are invariant under $\Phi = L_{\mathsf{a}}$, their Lie bracket $[X, Y]$ is also L_{a}-invariant. Letting a range over all of G, we thus conclude that the Lie bracket of two left-invariant vector fields is itself left-invariant.

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Thus, we have now shown that every Lie group G has an associated Lie algebra $\mathfrak{g}$, with underlying vector space $T_{\mathsf{e}}\mathsf{G}$.