Homework # 1

MAT 552

Due 3/19/21 at 11 pm

Do at least five problems. Extra credit for doing more!

1. A *topological group* is a topological space G that is also equipped with a group structure, in such a manner that the group multiplication

$$\begin{aligned} \mathsf{G} \times \mathsf{G} &\to \mathsf{G} \\ (\mathsf{a}, \mathsf{b}) &\mapsto \mathsf{ab} \end{aligned}$$

and the group inversion

$$\begin{aligned} \mathsf{G} &\to \mathsf{G} \\ \mathsf{a} &\mapsto \mathsf{a}^{-1} \end{aligned}$$

are both continuous maps. Notice that any Lie group is a topological group. We will denote the identity element of G by e.

Suppose that G is a locally-path-connected topological group, and let $\mathsf{G}_0 \subset \mathsf{G}$ be the connected component of G that contains the identity e. Notice that G_0 is also exactly the path-component of G that contains e. (Why?)

(a) Prove that G_0 is a normal subgroup of G.

(b) For any $\mathsf{a} \in \mathsf{G}$, show that the connected component of G that contains a is exactly the coset $\mathsf{a}G_0$. Then show that any two connected components of G are homeomorphic.

(c) Show that the set $\pi_0(\mathsf{G})$ of path components of G is exactly given by G/G_0, and therefore inherits a group structure from G. Also show that $\pi_0(\mathsf{G})$, equipped with the quotient topology, is a discrete topological group.

2. Let G be a locally-path-connected topological group. Prove that the fundamental group $\pi_1(\mathsf{G}, \mathsf{e})$ is Abelian. Then prove that the fundamental group $\pi_1(\mathsf{G}, \mathsf{a})$ relative to any other base-point $\mathsf{a} \in \mathsf{G}$ is canonically isomorphic to $\pi_1(\mathsf{G}, \mathsf{e})$, even if a does not happen to belong the identity component of G.

3. For any $n \geq 1$, we may embed the circle $\mathbf{U}(1) \subset \mathbb{C}^\times$ into the unitary group $\mathbf{U}(n)$ as the scalar multiples

$$e^{it}\mathbf{I} = \begin{bmatrix} e^{it} & & & \\ & e^{it} & & \\ & & \ddots & \\ & & & e^{it} \end{bmatrix}, \qquad t \in \mathbb{R},$$

of the identity matrix by complex numbers of unit modulus.

(a) Prove that this circle $\mathbf{U}(1) \subset \mathbf{U}(n)$ is exactly the center Z of $\mathbf{U}(n)$, where the center Z of a group G is defined by

$$Z(\mathsf{G}) := \{\mathsf{a} \in \mathsf{G} \mid \mathsf{ab} = \mathsf{ba} \quad \forall\ \mathsf{b} \in \mathsf{G}\}.$$

(b) Show that this circle $\mathbf{U}(1) \subset \mathbf{U}(n)$ is the image of a specific 1-dimensional linear subspace of $\mathfrak{u}(n)$ under $\exp : \mathfrak{u}(n) \to \mathbf{U}(n)$. Describe this subspace, and show that it is exactly the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{u}(n)$, where the center $\mathfrak{z}$ of a Lie algebra $\mathfrak{g}$ is defined by

$$\mathfrak{z}(\mathfrak{g}) := \{\mathbf{X} \in \mathfrak{g} \mid [\mathbf{X}, \mathbf{Y}] = 0 \quad \forall\ \mathbf{Y} \in \mathfrak{g}\}.$$

Hint. If two linear transformations commute, how are their eigenspaces related? Now observe that there are elements of $\mathbf{U}(n)$ and $\mathfrak{u}(n)$ that have any given 1-dimensional complex-linear-subspace $\mathbb{L} \subset \mathbb{C}^n$ as an eigenspace.

4. Use Problem 3 to show that the center of $\mathbf{SU}(n)$ is contained in the center of $\mathbf{U}(n)$. Then use this to show that the center of $\mathbf{SU}(n)$ is exactly

$$Z(\mathbf{SU}(n)) = \mathbf{U}(1) \cap \mathbf{SU}(n) \cong \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}.$$

5. (a) Use Problem 4 to show that, for each n, there is a compact connected Lie group G with $\pi_1(\mathsf{G}, \mathsf{e}) \cong \mathbb{Z}_n$.

(b) Recall that the Cartesian product of two Lie groups is also a Lie group. Also remember that the circle $\mathbf{U}(1)$ is a compact connected Lie group. By combining these facts with part (a), prove that any finitely generated Abelian group Γ is isomorphic to the fundamental group $\pi_1(\mathsf{G}, \mathsf{e})$ of some compact connected Lie group G.

(c) The fundamental group of any compact manifold is finitely generated. Given this, show that a group Γ is the fundamental group of some compact connected Lie group if and only if Γ is finitely generated and Abelian.

6. A Lie-group homomorphism is by definition a smooth map between Lie groups that is also a group homomorphism. By taking the derivative of such a smooth map at the identity, show that any Lie group homomorphism $\Phi : \mathsf{G} \to \mathsf{H}$ induces a homomorphism of the corresponding Lie algebras, which is to say a linear map $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ such that

$$[\Phi_* \mathbf{X}, \Phi_* \mathbf{Y}] = \Phi_* [\mathbf{X}, \mathbf{Y}] \quad \forall \mathbf{X}, \mathbf{Y} \in \mathfrak{g}.$$

7. Let G be a connected Lie group, and let $U \subset \mathsf{G}$ be an open neighborhood of the identity $\mathsf{e} \in \mathsf{G}$. Show that U *generates* G, in the sense that any $\mathsf{a} \in \mathsf{G}$ is a finite product $\mathsf{a}_1 \mathsf{a}_2 \cdots \mathsf{a}_k$ of elements $\mathsf{a}_j \in U$.

8. Let G and H be connected Lie groups. Prove that a Lie-group homomorphism $\Phi : \mathsf{G} \to \mathsf{H}$ is a covering map (as a map between smooth manifolds) iff $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras.

9. Prove that

$$\begin{array}{rcl} \Phi : \mathbf{U}(1) \times \mathbf{SU}(n) & \to & \mathbf{U}(n) \\ (e^{it} \quad , \quad A) & \mapsto & e^{it} A \end{array}$$

is an n-to-1 covering map. Then prove that $\mathbf{U}(1) \times \mathbf{SU}(n)$ and $\mathbf{U}(n)$ are however not isomorphic as Lie groups.

10. Let G be the 2-torus $\mathbf{U}(1) \times \mathbf{U}(1)$, and let $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \cong \mathbb{R}^2$ be its Lie algebra. Let $\mathbf{X} \in \mathfrak{g}$, and let H be the closure of $\{\exp(t\mathbf{X}) \mid t \in \mathbb{R}\} \subset \mathsf{G}$. Prove that H is either a point, an embedded circle, or the entire 2-torus G. Show that each of these possibilities actually occurs, and state a concrete criterion that predicts which one will arise from a given $\mathbf{X}$.