MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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Stony Brook University

May 7, 2020

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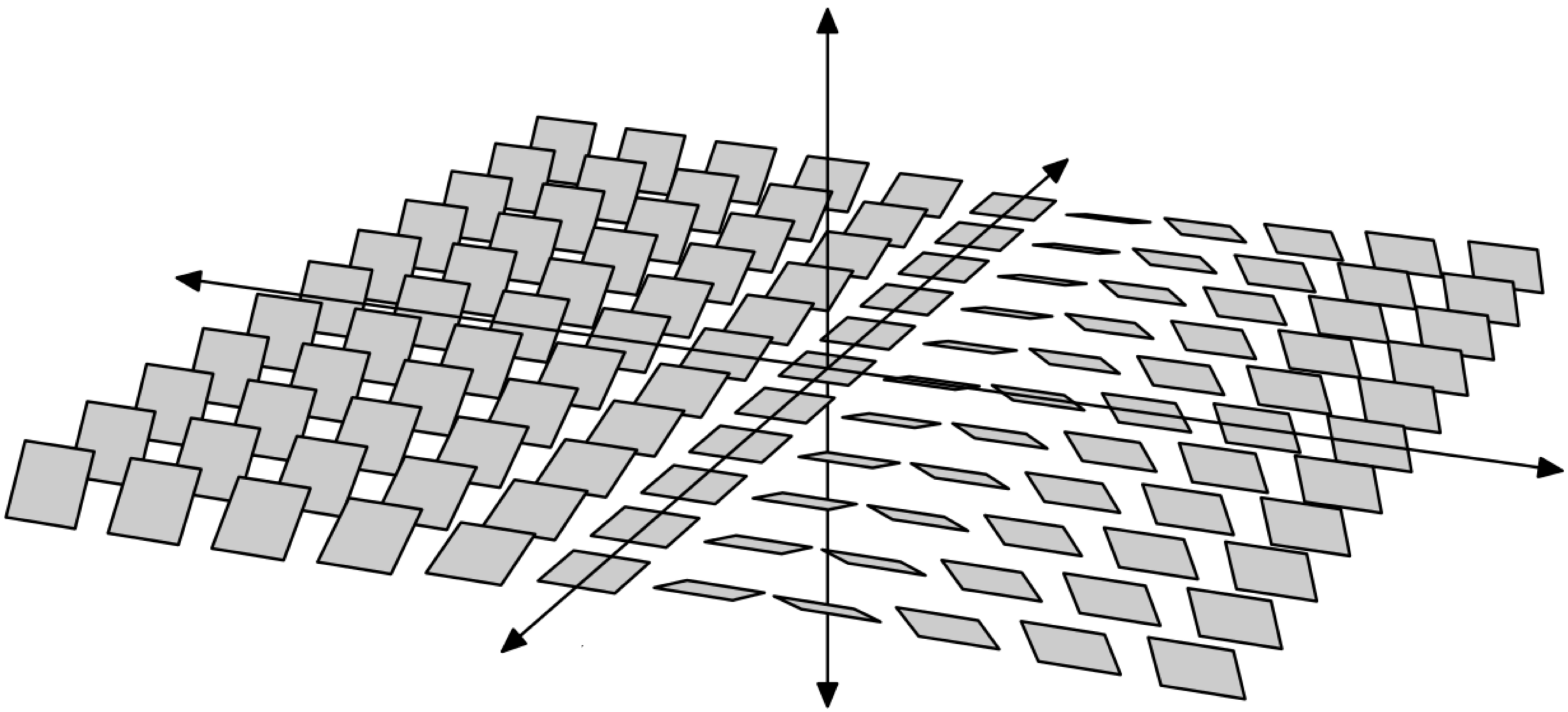

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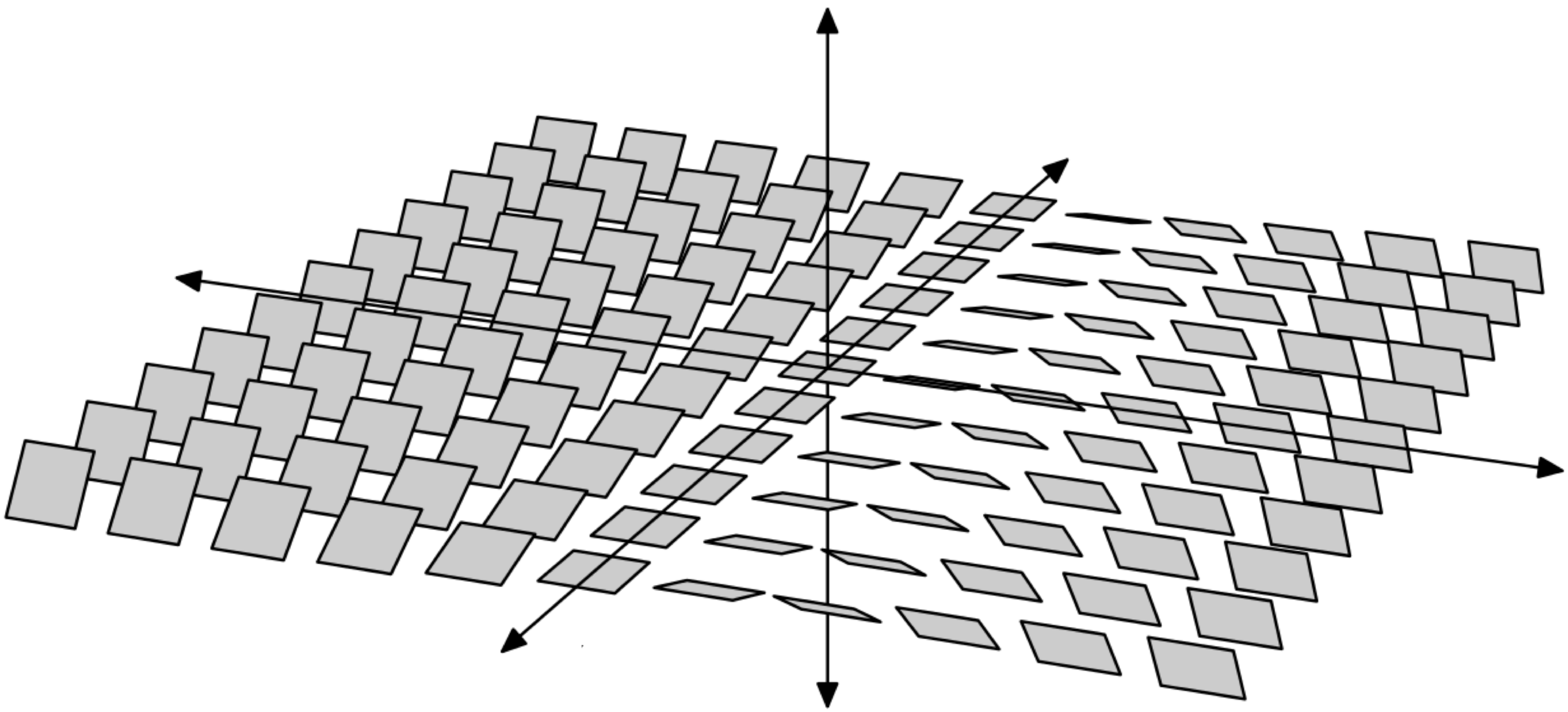

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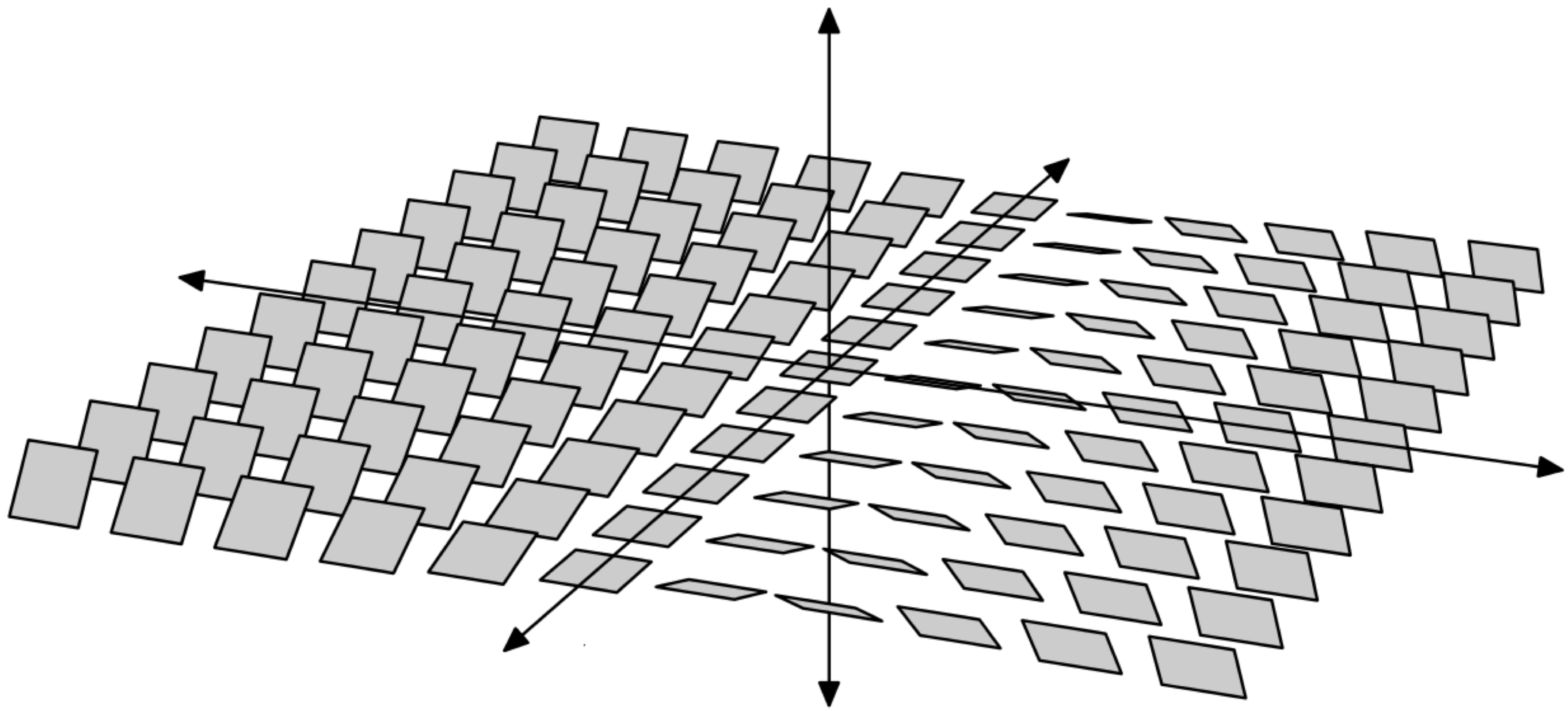

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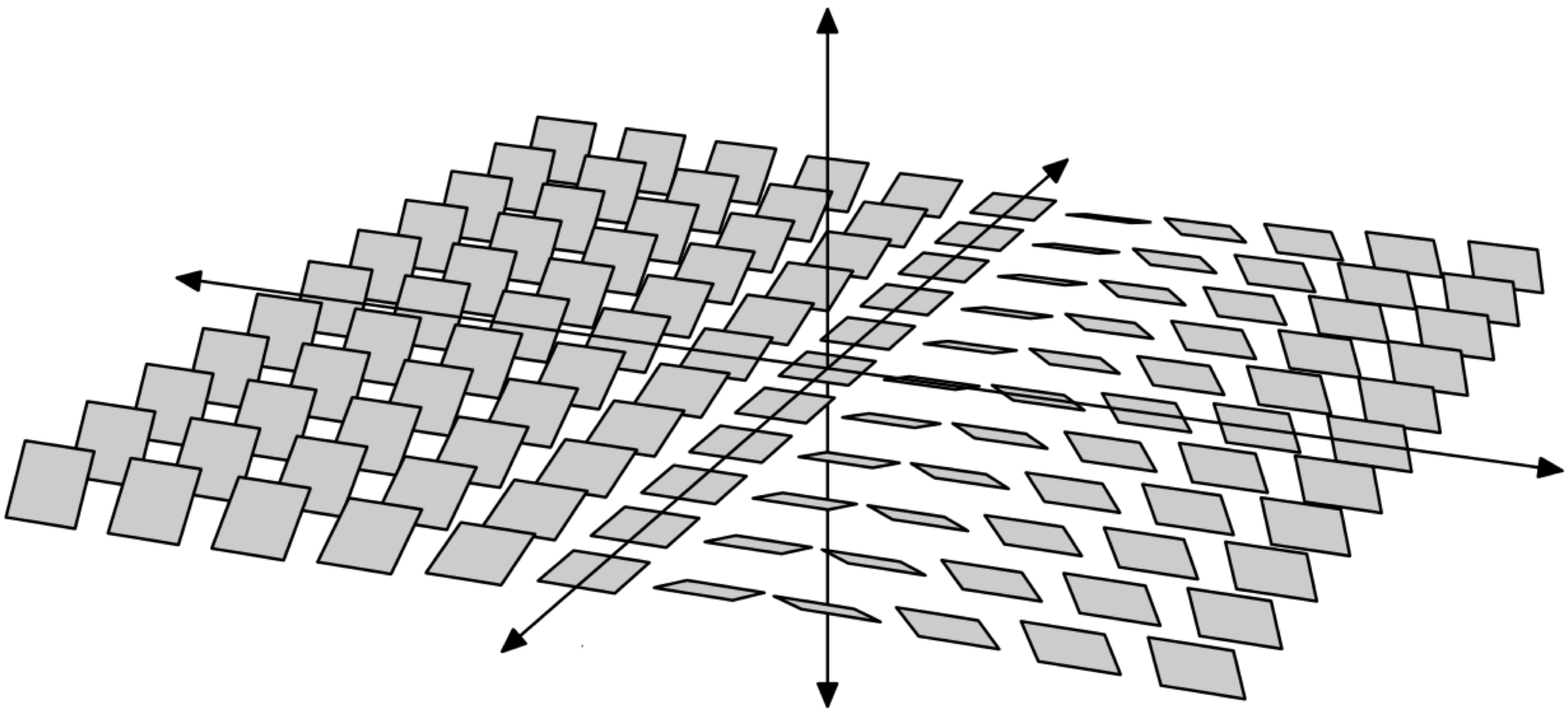

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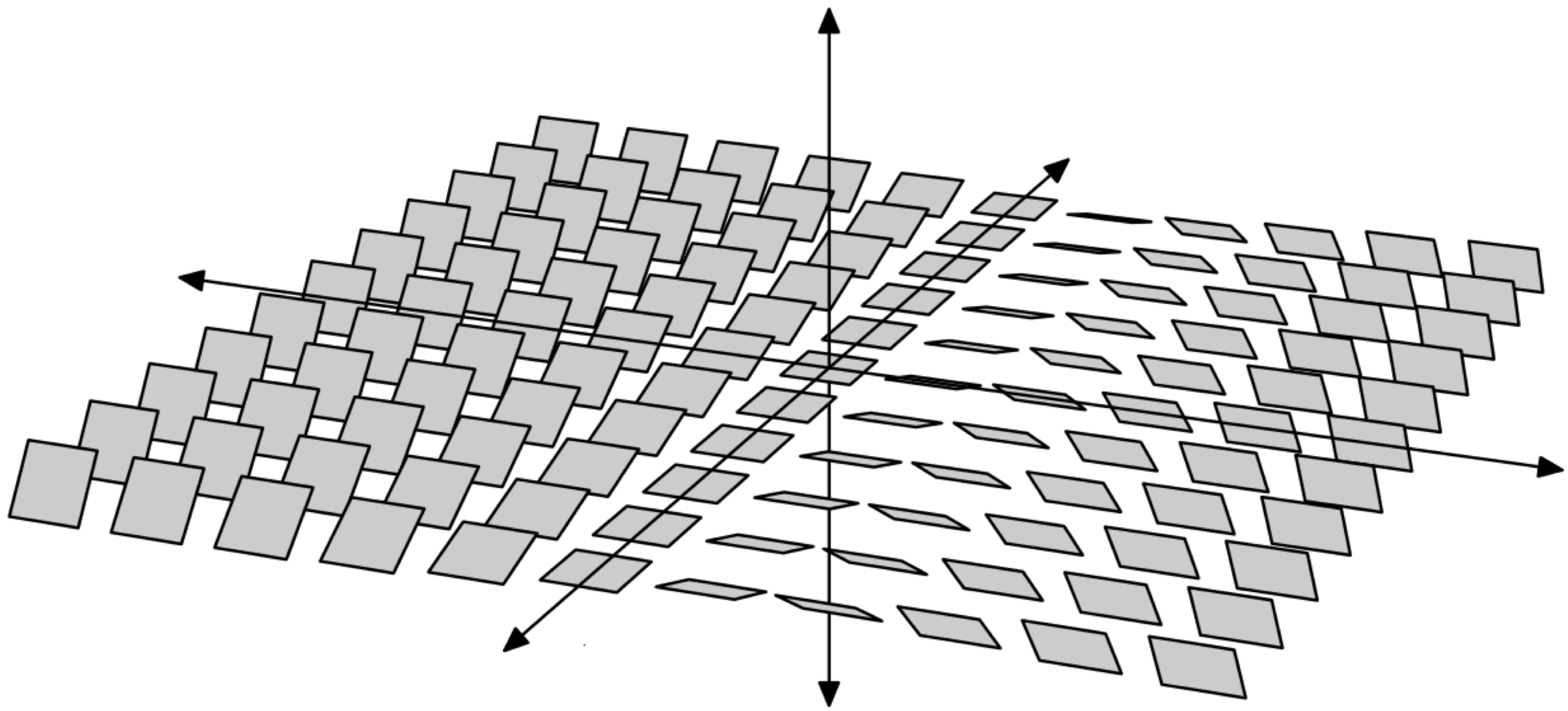

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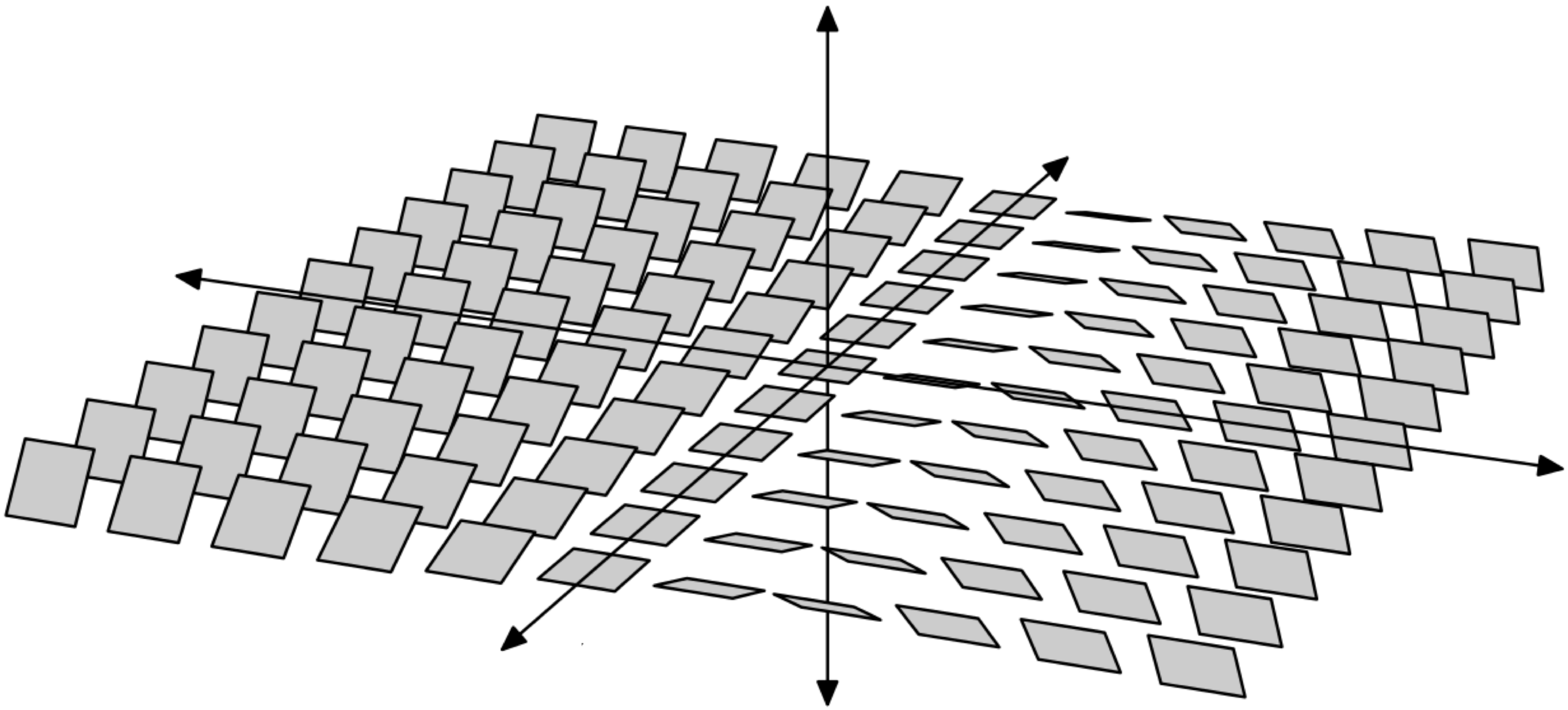

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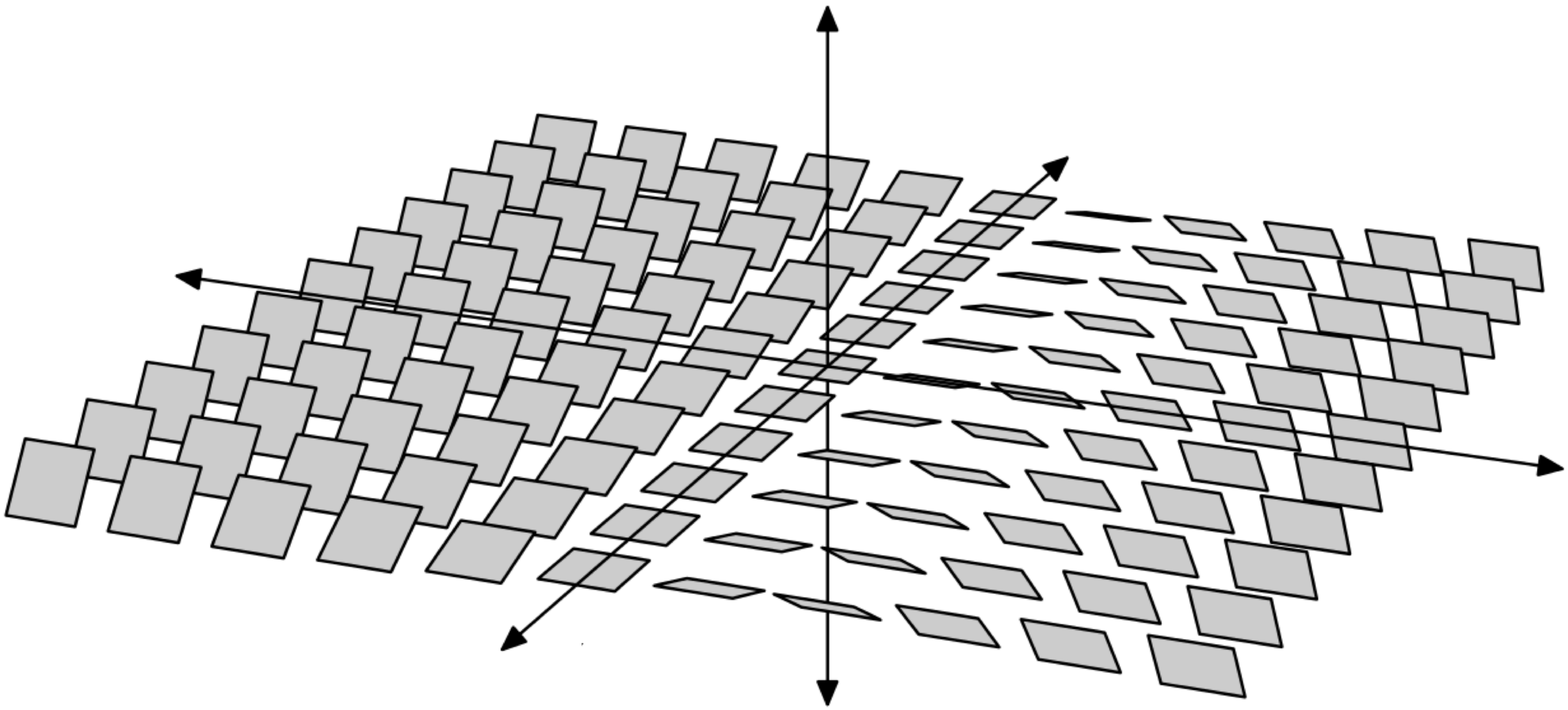

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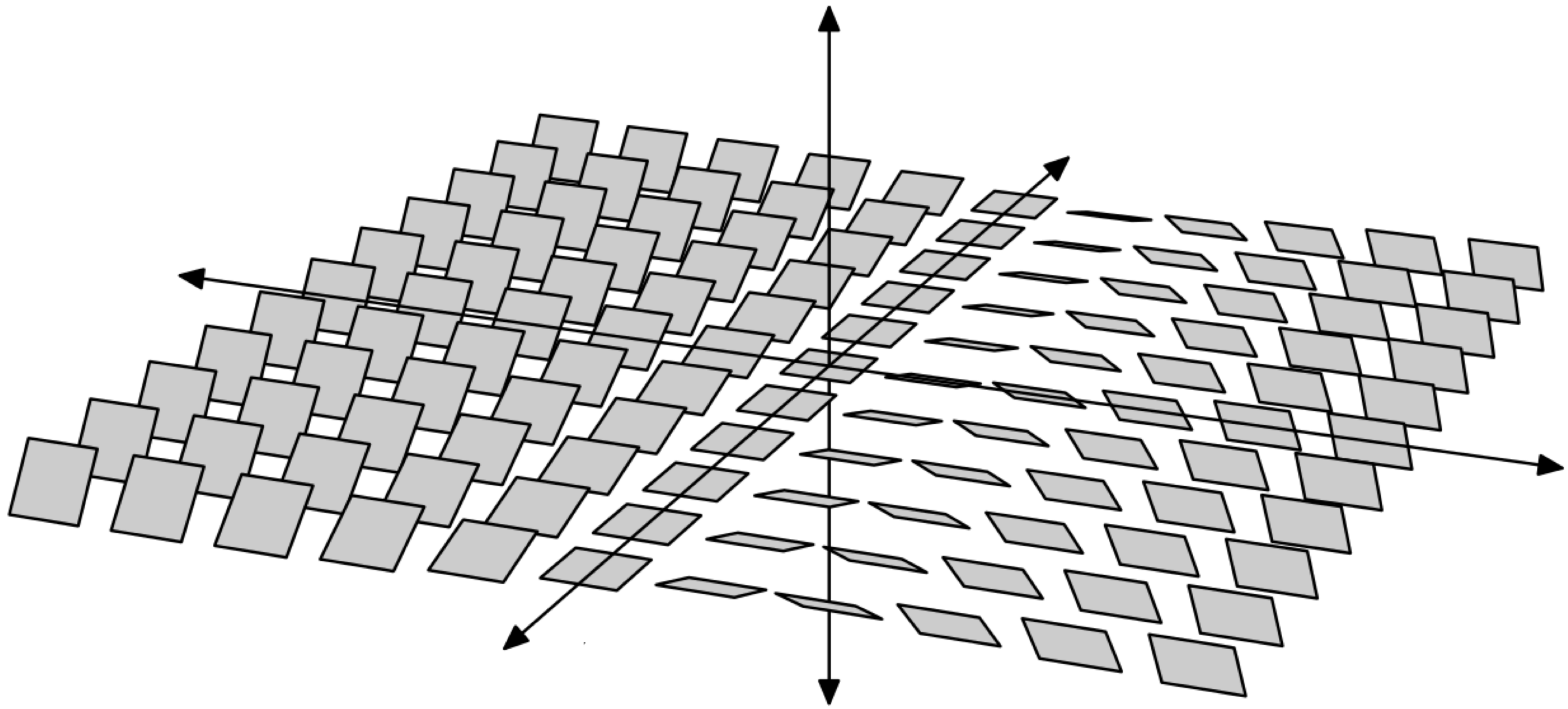

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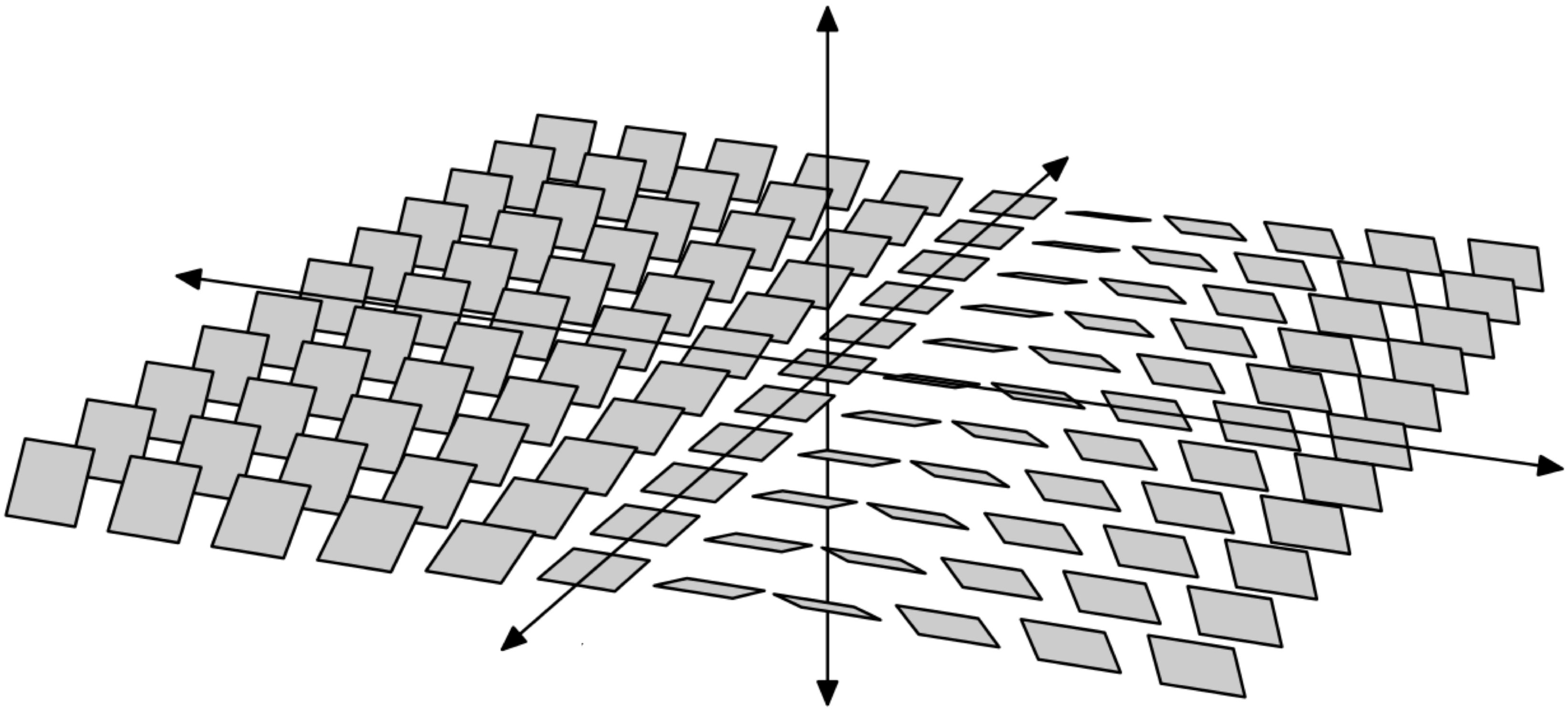

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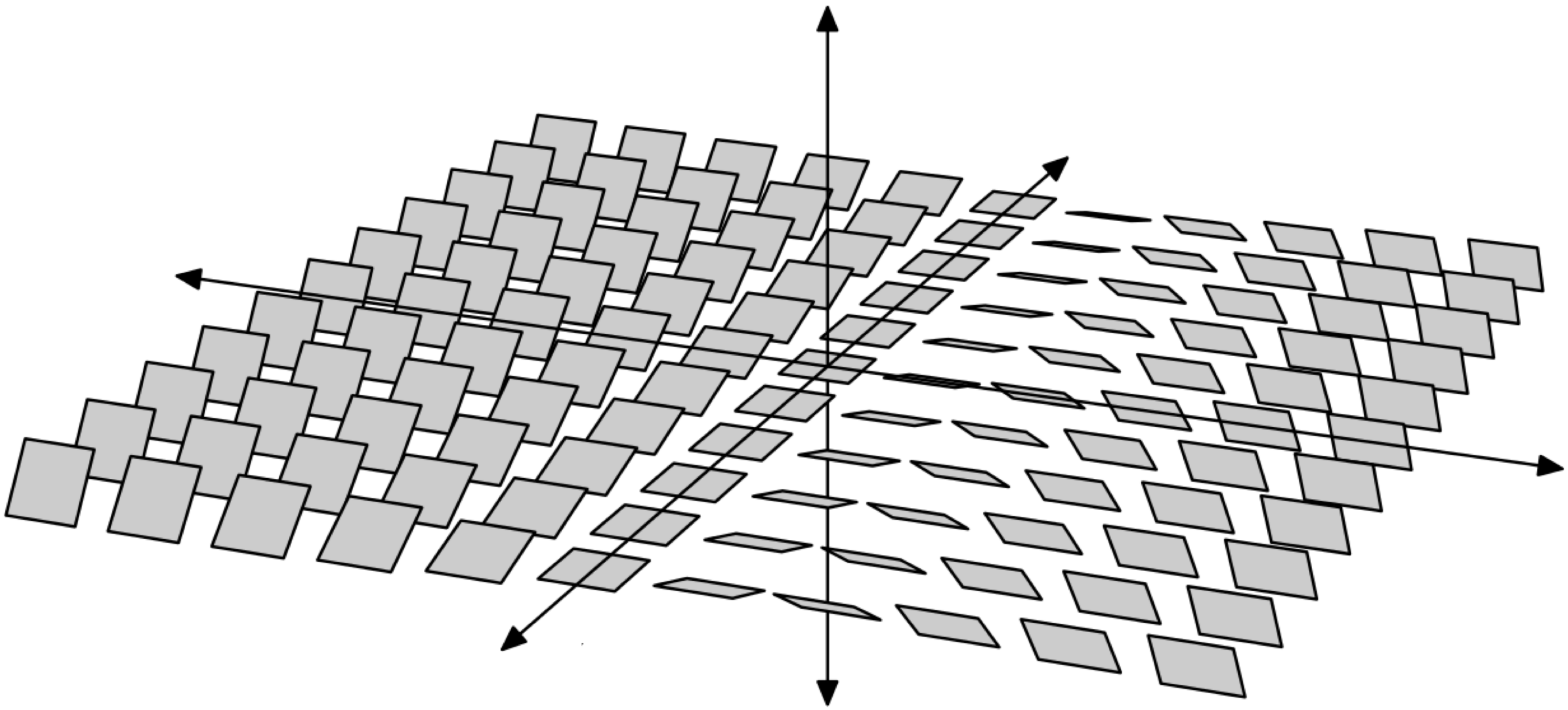

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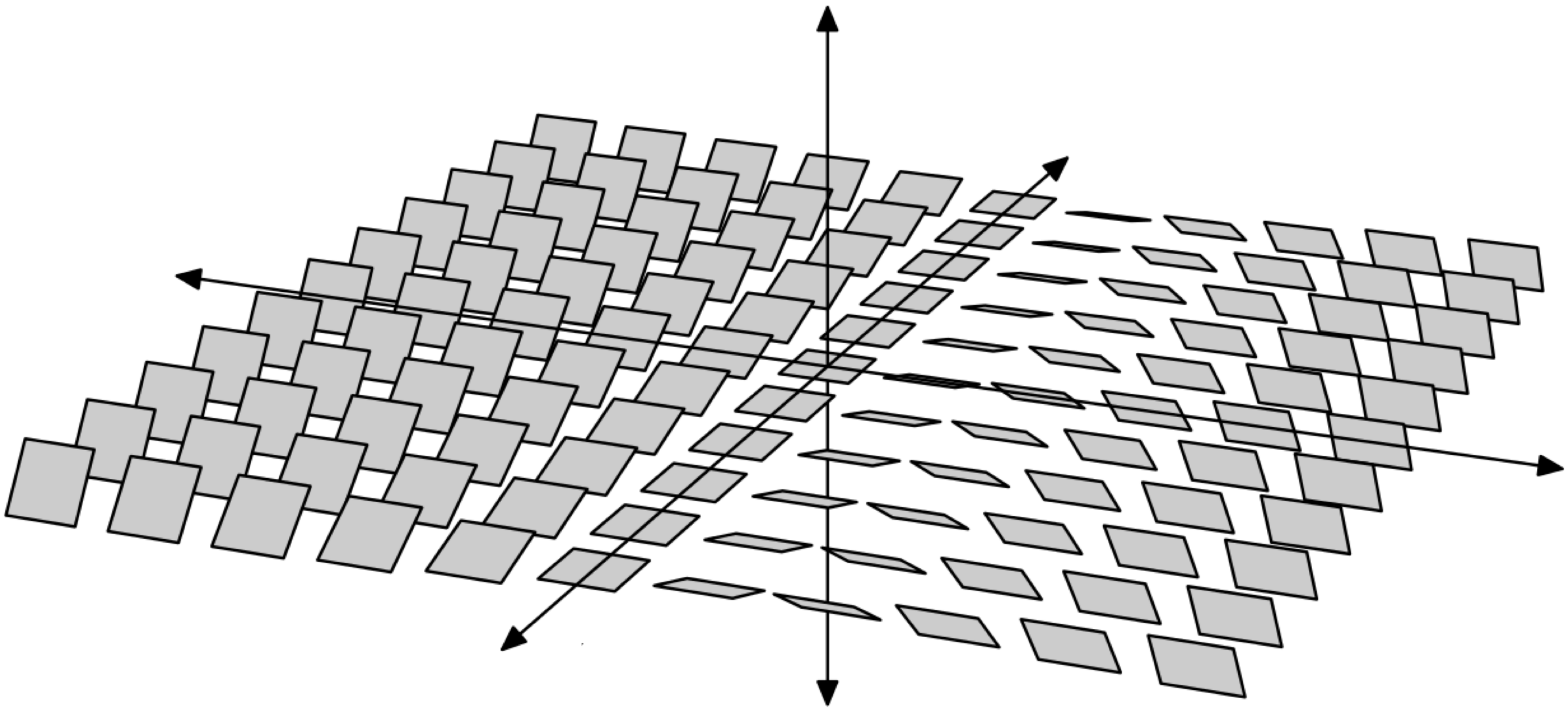

Any such Σ is called an integral manifold of D.

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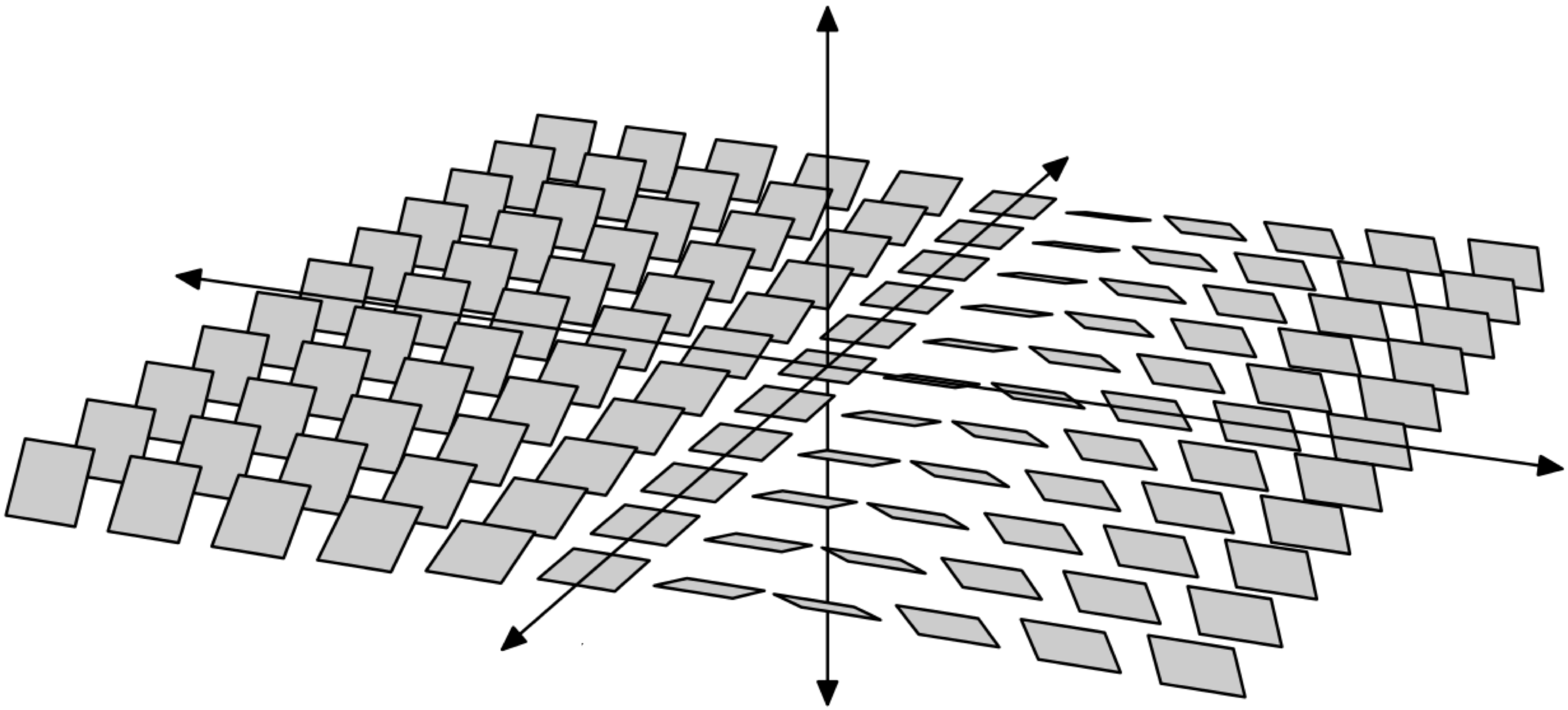

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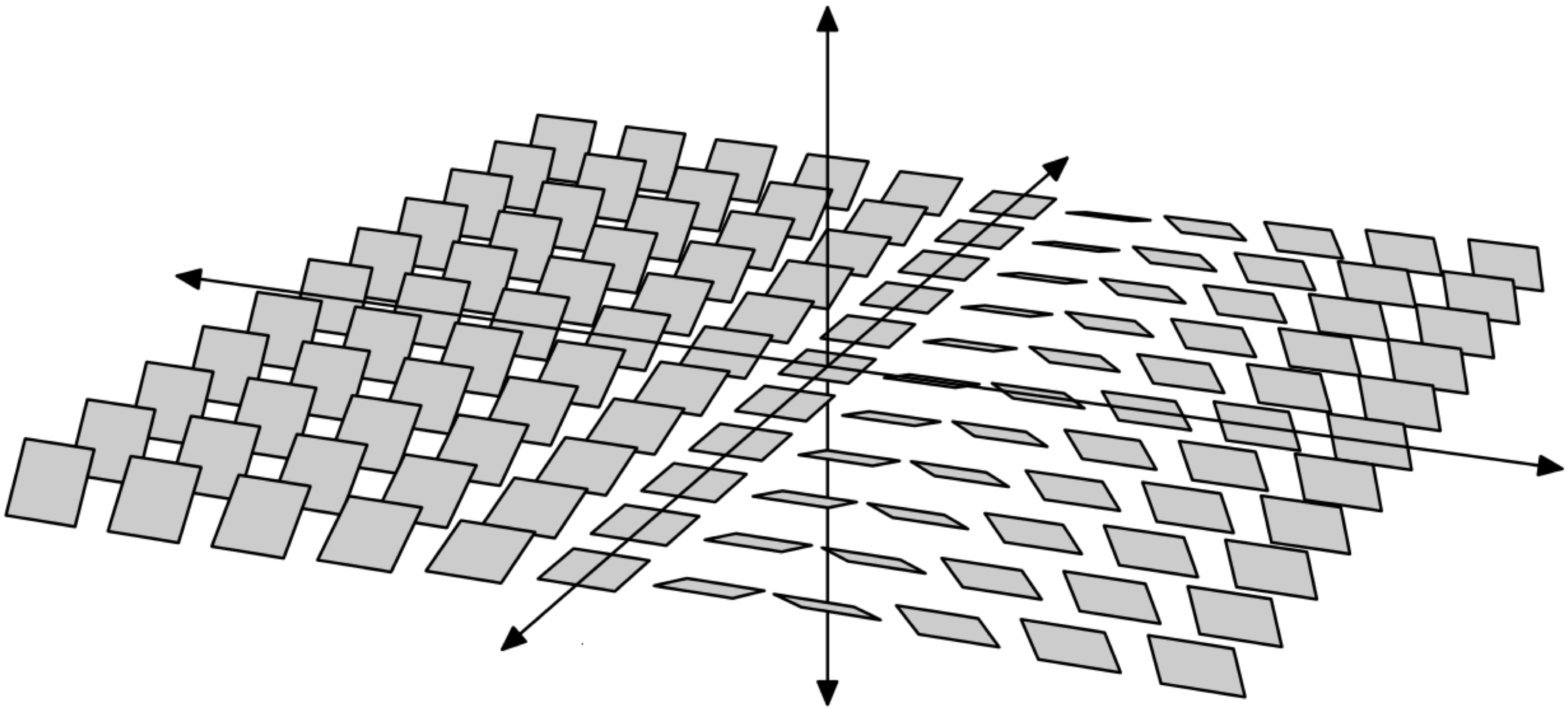

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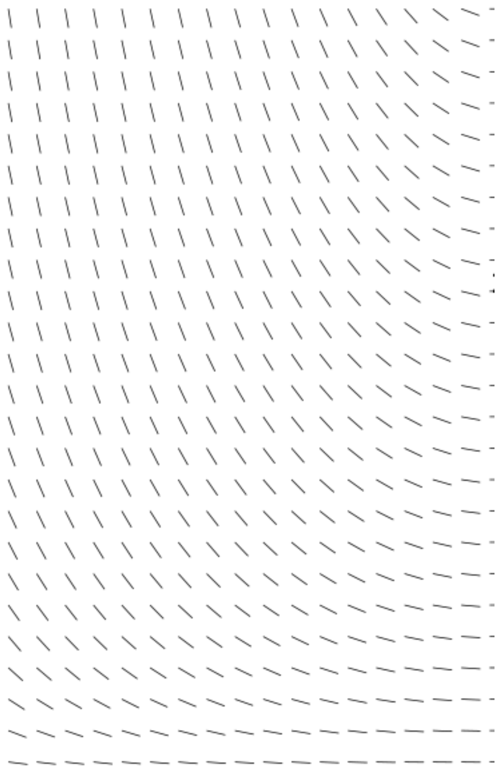

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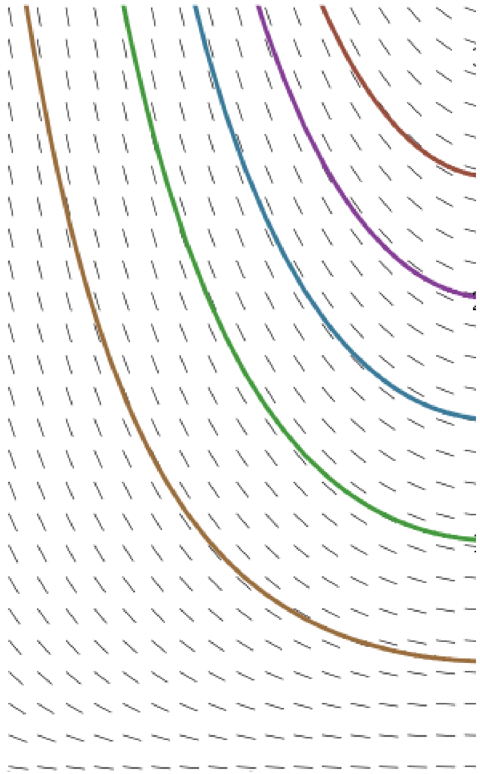

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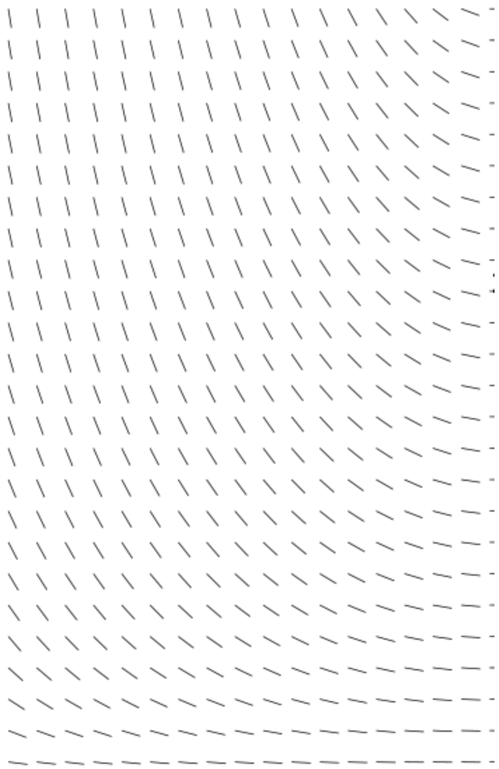

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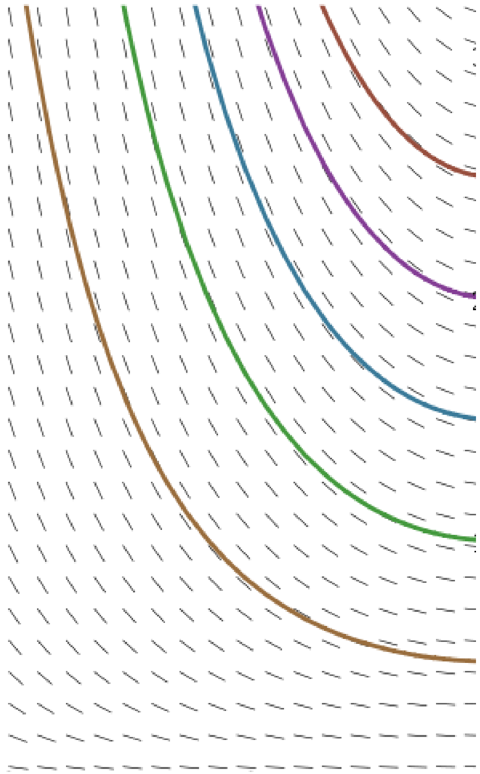

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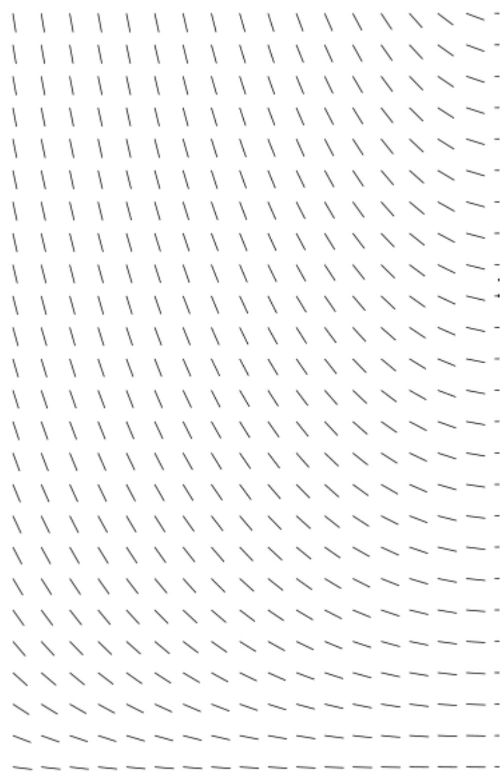

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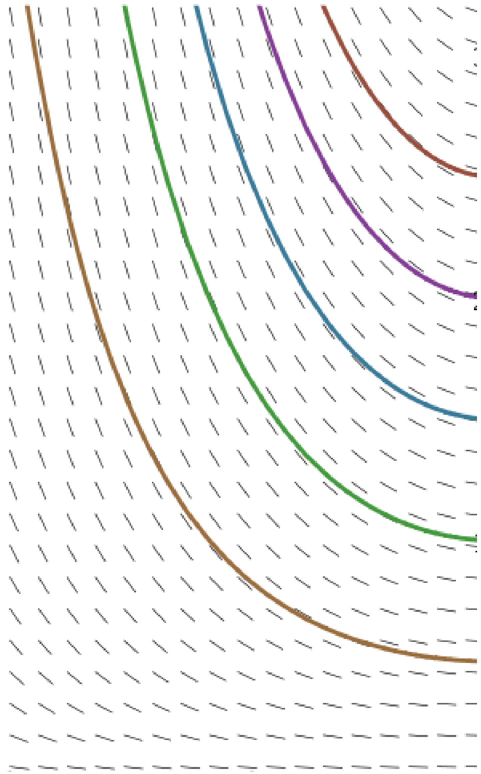

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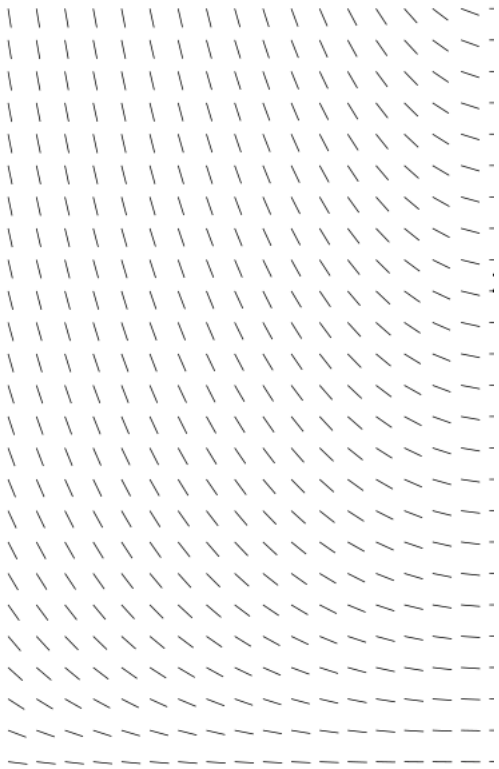

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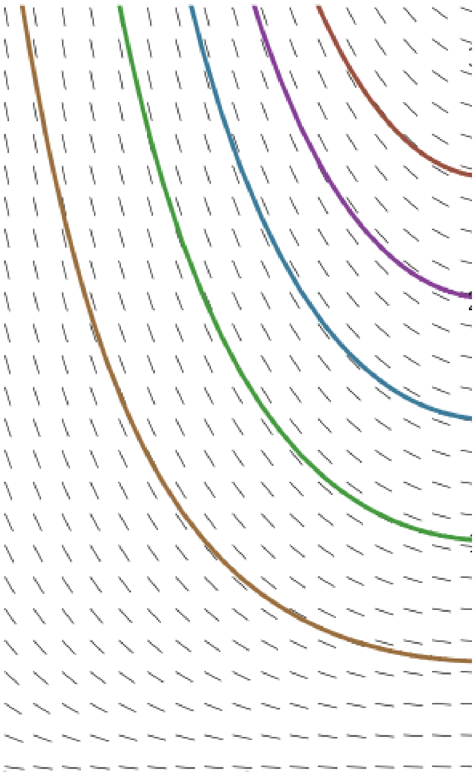

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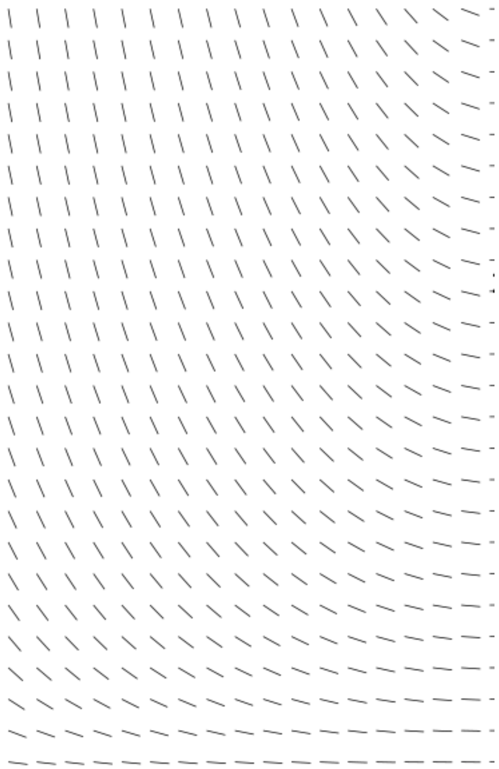

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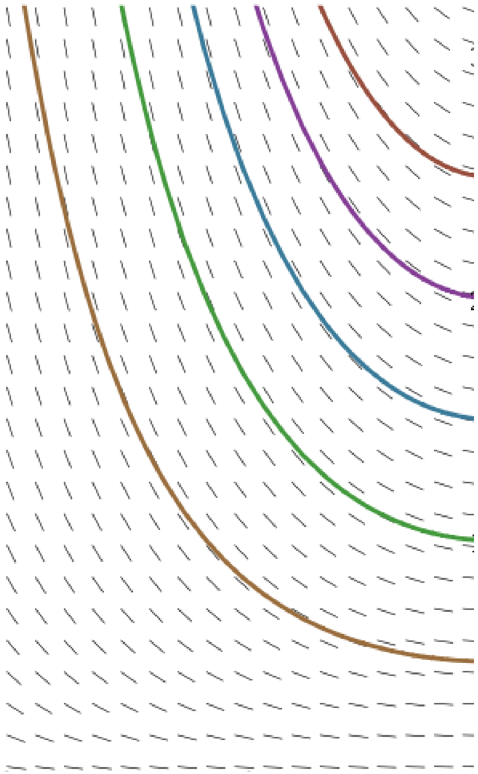

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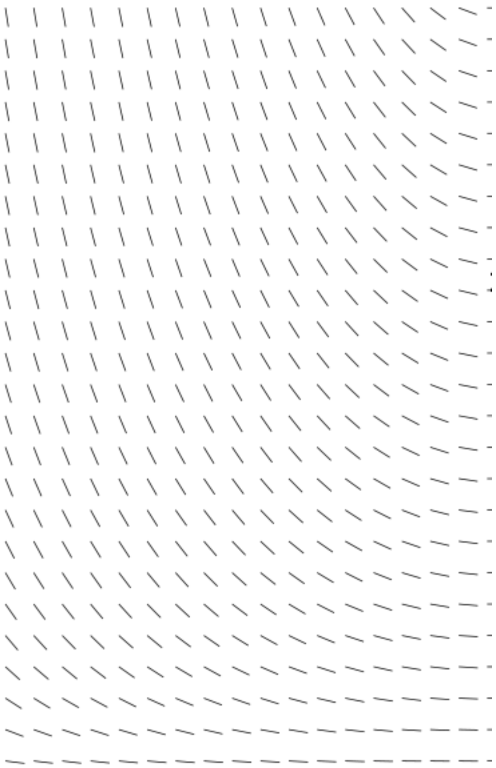

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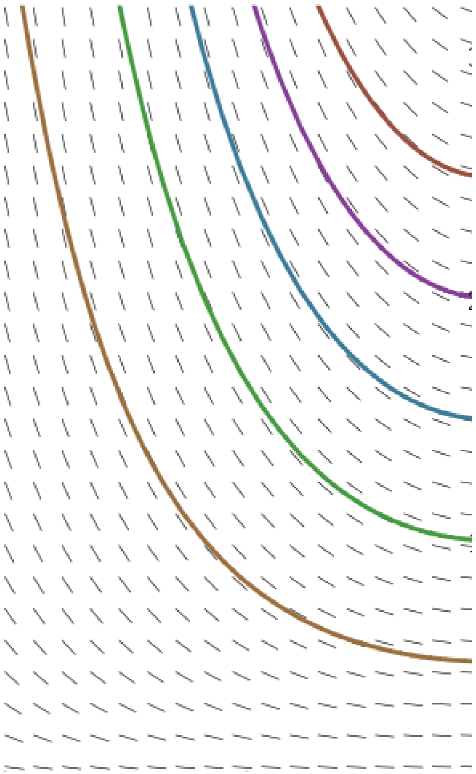

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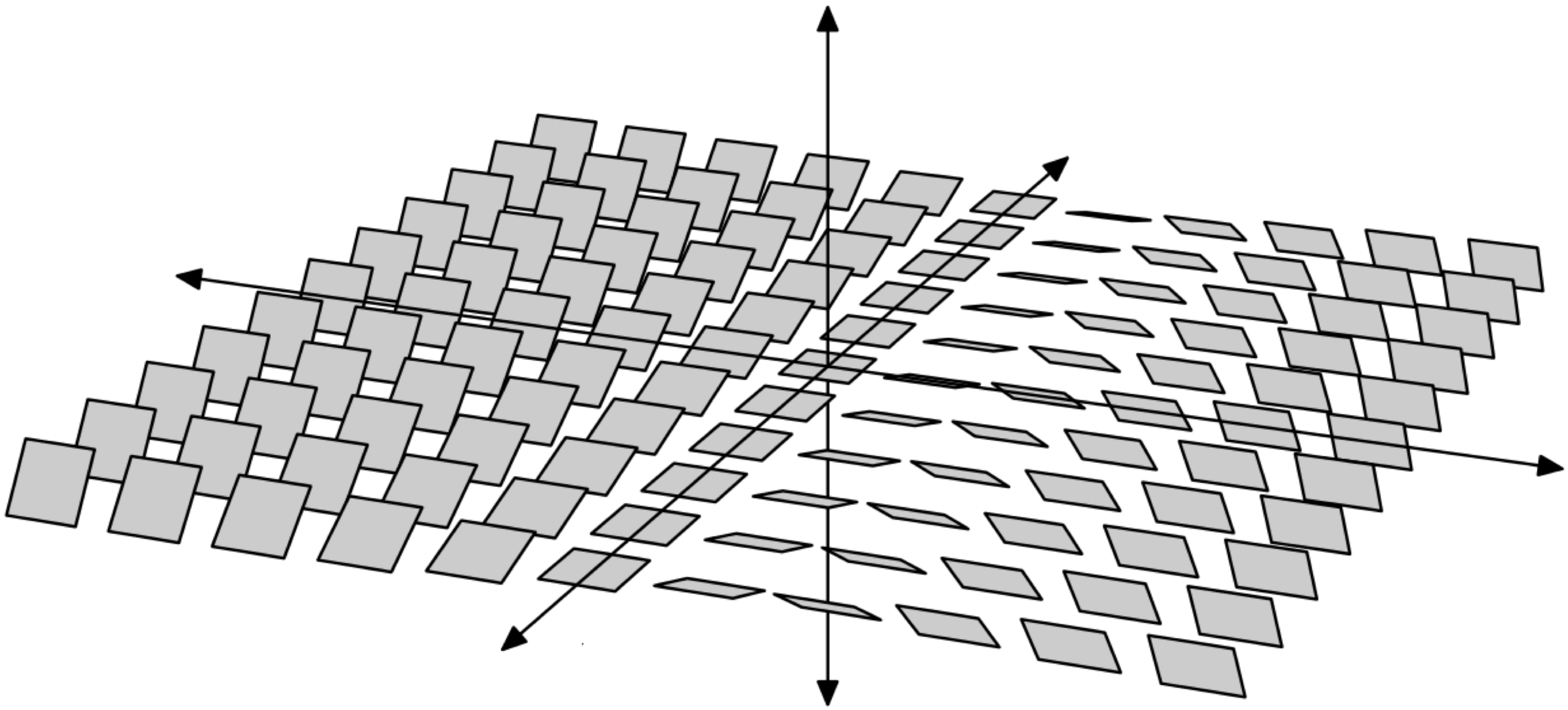

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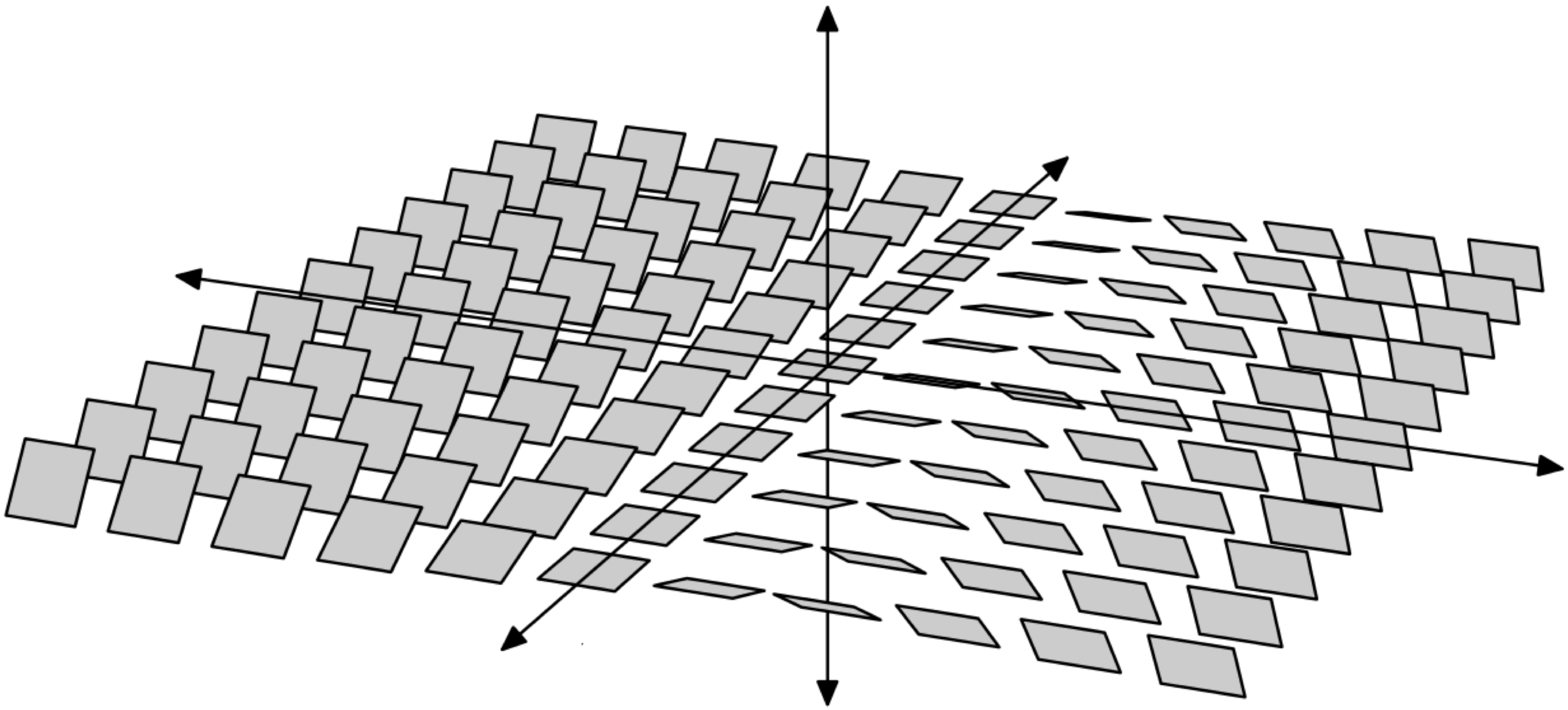

But when $\ell > 1$, another issue comes into play...

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$$[V_i, V_j] \in \text{span}\,\{V_1, \ldots, V_\ell\}$$

Basic Problem:

Are there ℓ-dimensional submanifolds $\Sigma \subset M$ such that D is tangent to Σ at each of its points?

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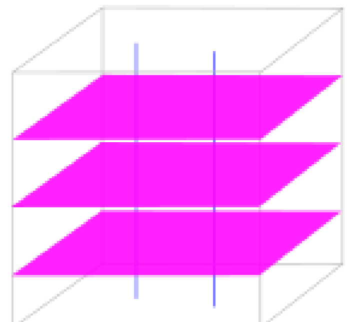
everywhere on M. *Moreover, when this happens, can find coordinates* $(x^1, \ldots, x^n)$ *near any* $p \in M$ *in which*

$$D = span\,\{\frac{\partial}{\partial x^1}, \ldots \frac{\partial}{\partial x^\ell}\}.$$



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