MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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Closed Forms.

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De Rham complex:

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Makes sense for manifolds-with-boundary, too.

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$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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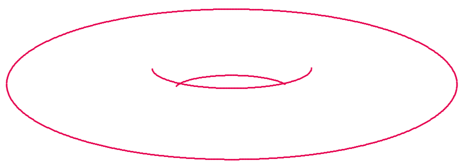

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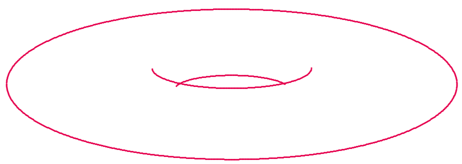

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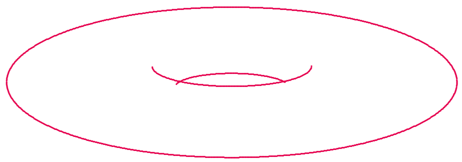

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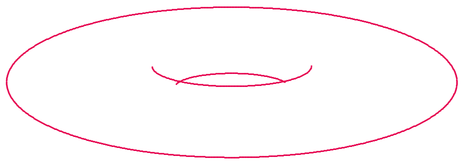

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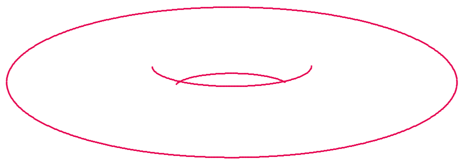

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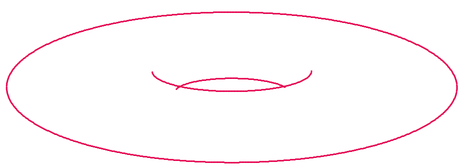

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$\therefore$ If $\varphi \in \Omega^n(M)$ has $\int_M \varphi \neq 0$, then $\varphi \neq d\eta$.

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That is, an n-form is exact iff its integral $= 0$.

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Same conclusion if M compact manifold-with-boundary, where $\partial M \neq \varnothing$.

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& \uparrow F^* & & \uparrow F^* & & \uparrow F^* & \\
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$$H^k(M) \xleftarrow{F^*} H^k(N)$$

$$M \xrightarrow{F} N$$

Compactly supported cohomology.

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Let $\Omega_c^k(M) = \{\text{compactly supported smooth } k\text{-forms}\}$.

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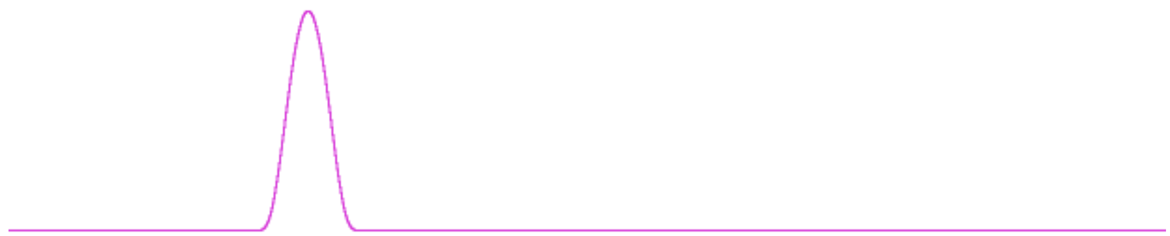

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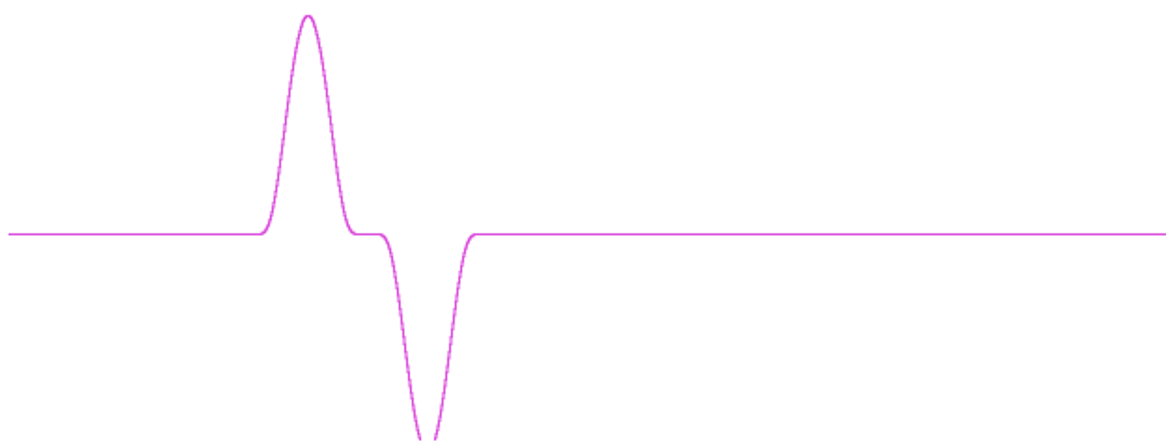

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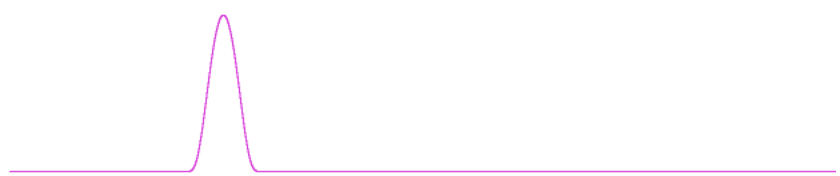

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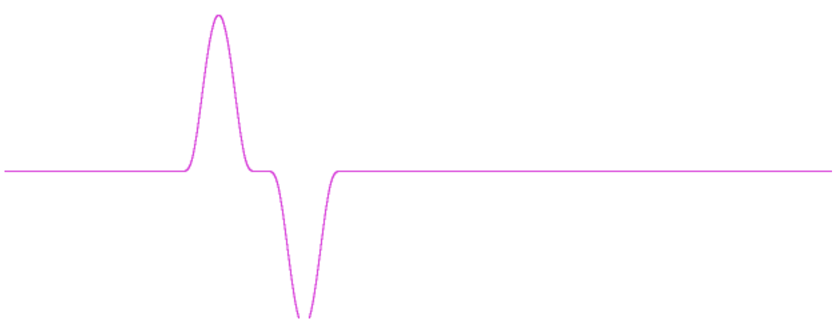

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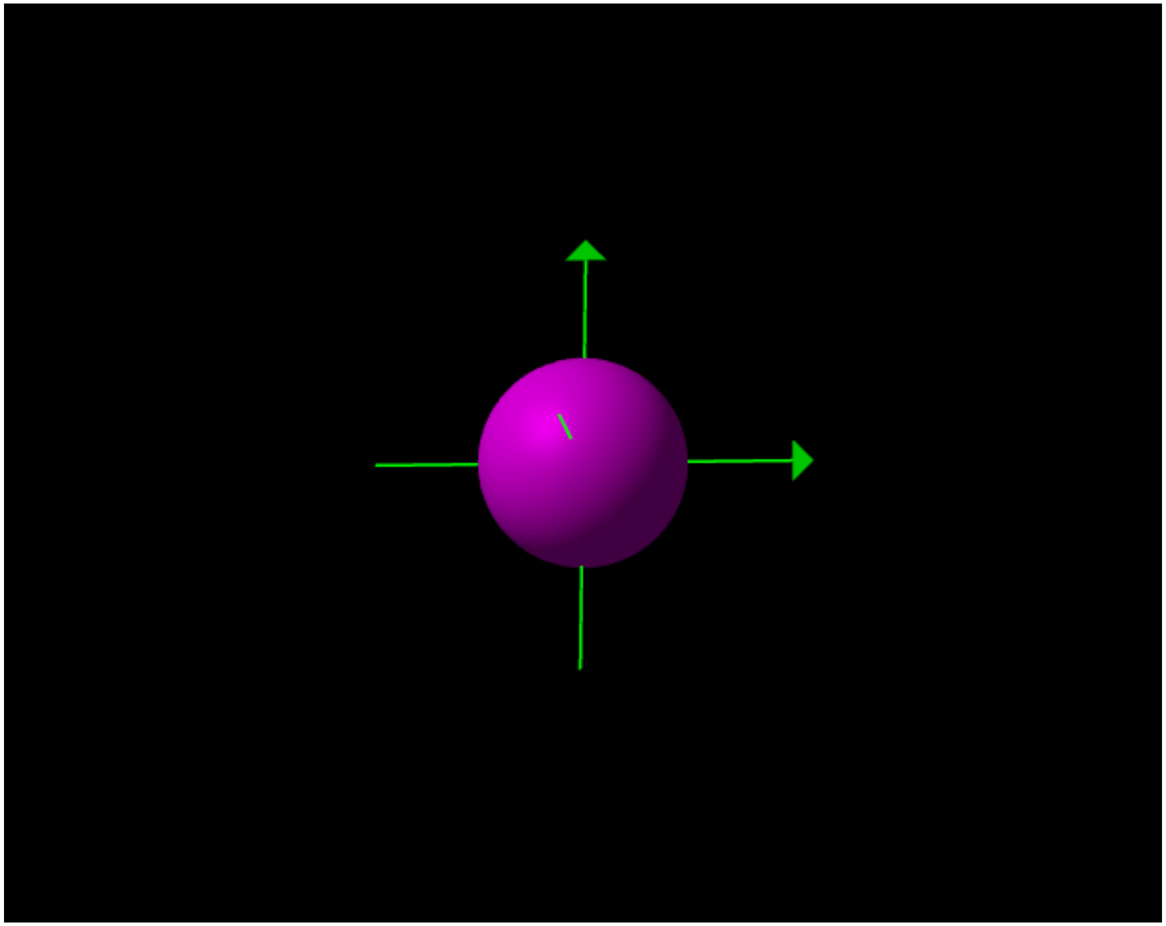

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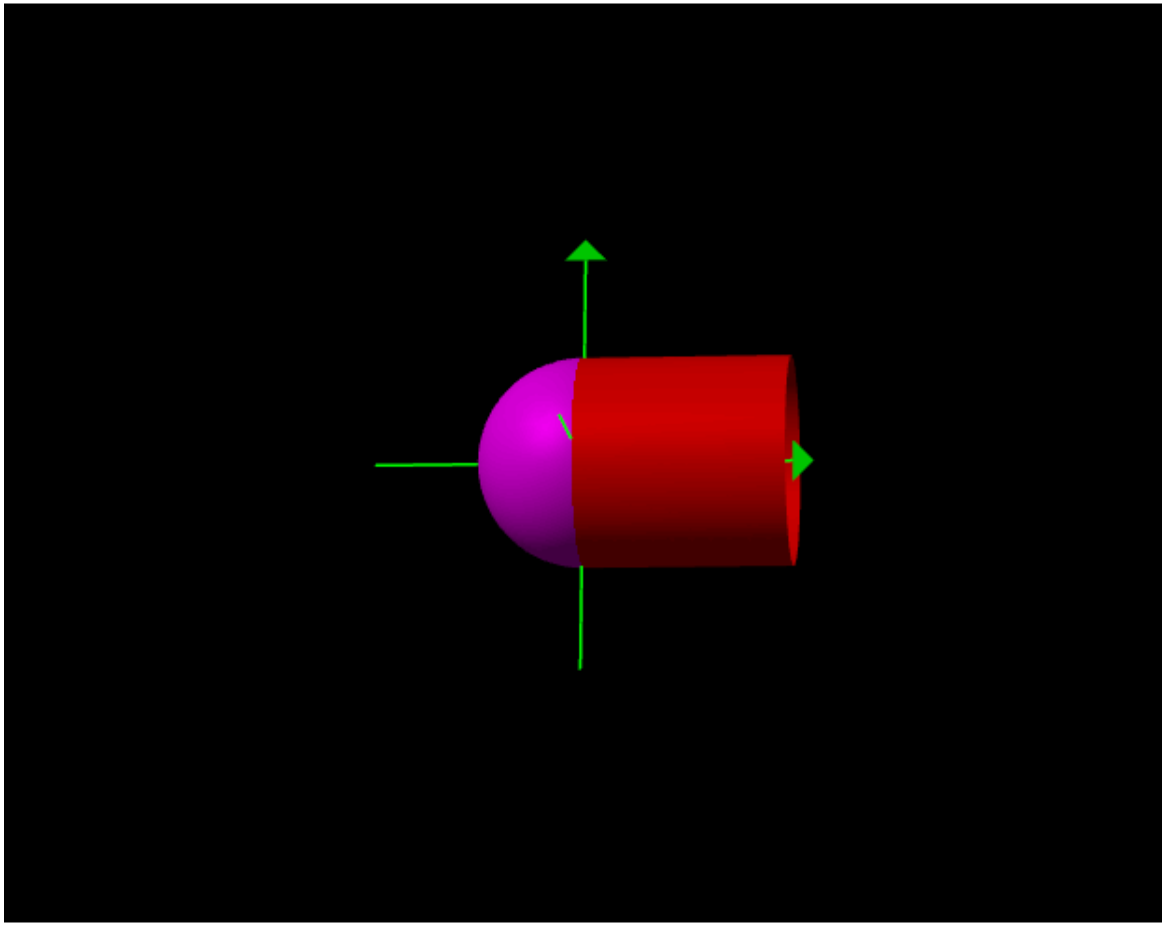

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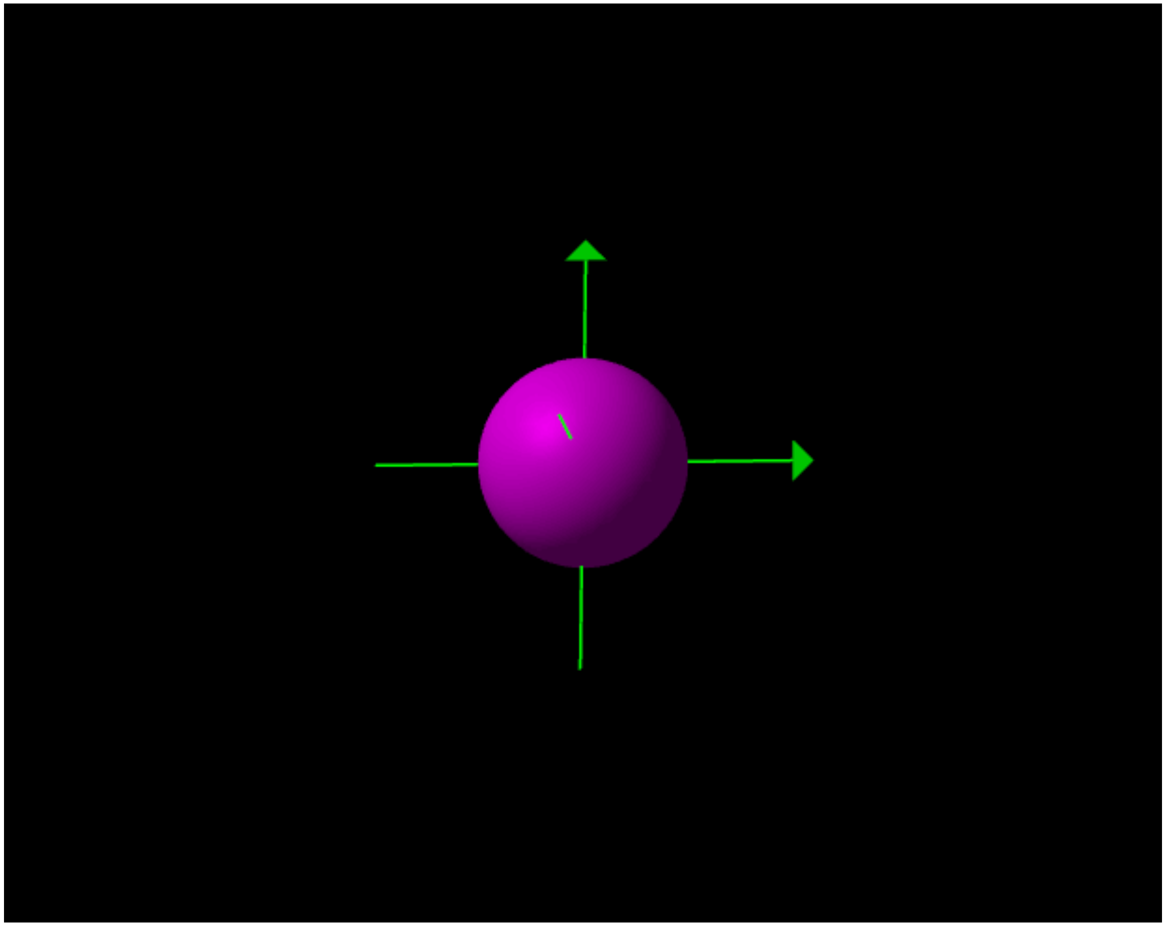

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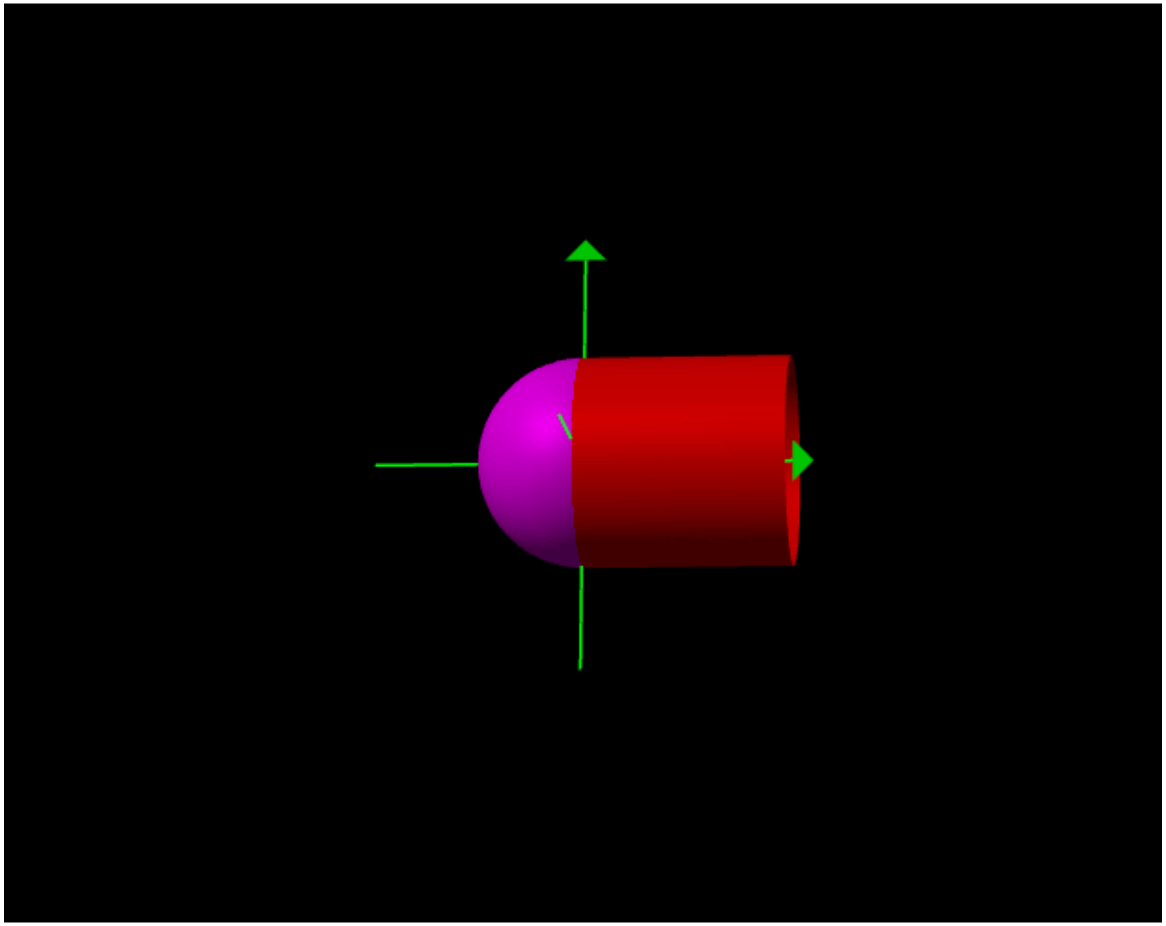

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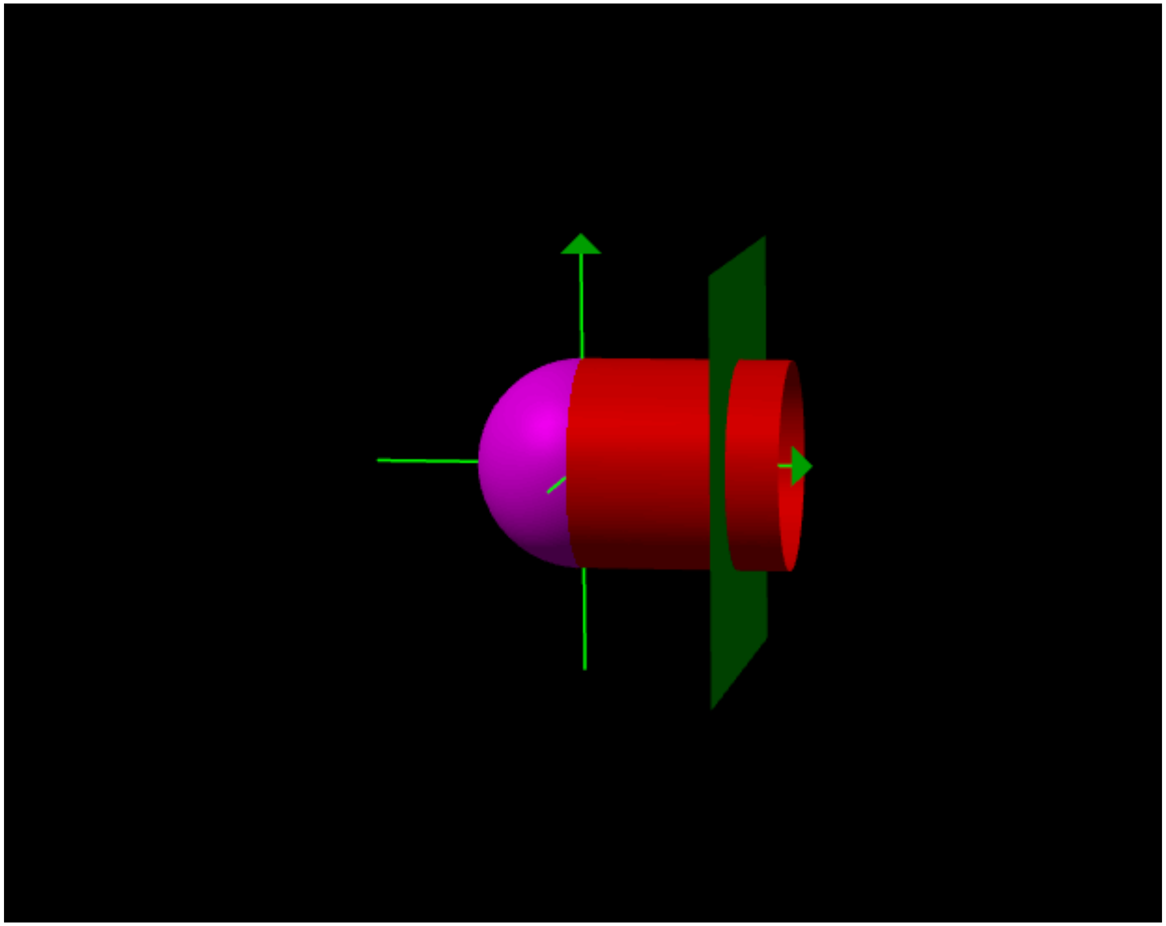

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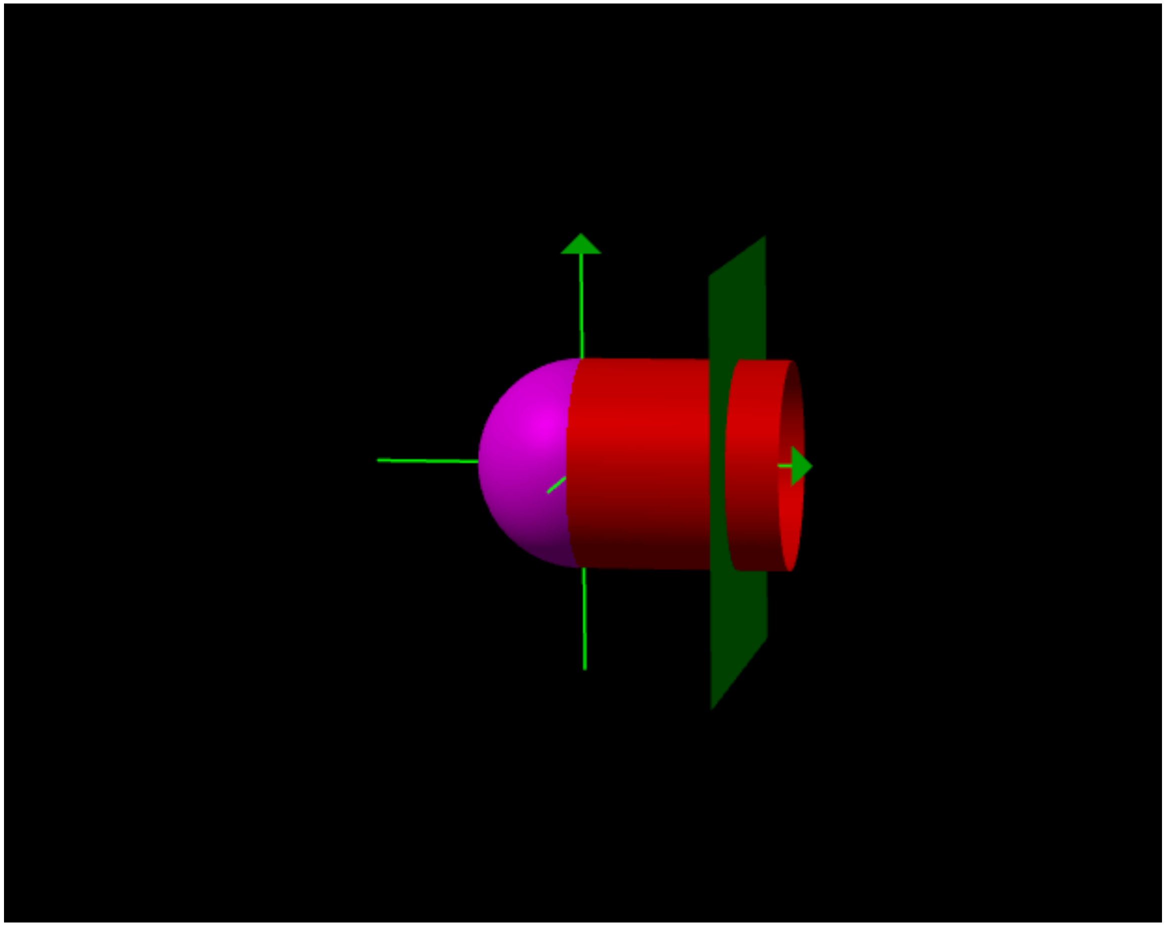

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Using this, we now prove a major generalization. . .

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Now recall an application of flow of a vector field. . .

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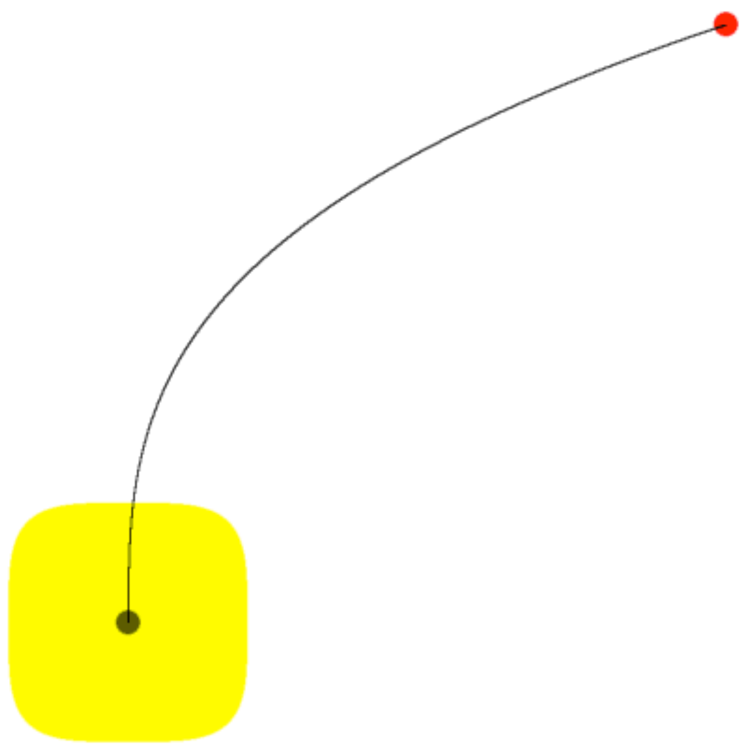
Lemma. *Let M^n be a smooth connected n-manifold, and let $p, q \in M$ be any two points. Then M contains a coordinate domain $\mathscr{U} \approx \mathbb{R}^n$ such that $p, q \in \mathscr{U}$.*



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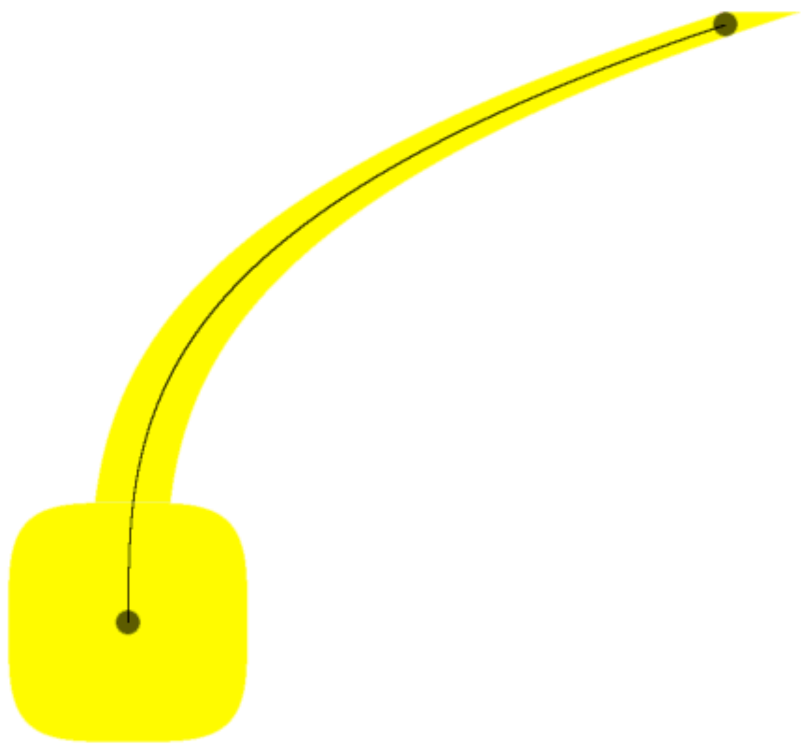

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Then each n-forms φ_j is then compactly supported in $\mathscr{U}_j$, and

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This now implies...

Theorem. *If M^n is a connected, oriented smooth n-manifold (without boundary), then*

$$H_c^n(M^n) \cong \mathbb{R},$$

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Theorem. *If M^n is a connected, oriented smooth n-manifold (without boundary), then*

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Specializing to the compact case, we thus have...

Theorem. *If* M^n *is a smooth compact, connected, oriented* n*-manifold (without boundary), then*

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