MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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Lie derivatives:

Lie derivative of tensor field φ w/resp. to V:

Lie derivatives:

φ

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$$\Phi_t^* \varphi$$

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$$\frac{d}{dt}\Phi_t^*\varphi$$

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$$\frac{d}{dt}\Phi_t^*\varphi\Big|_{t=0}$$

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But there is a more efficient formula!

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Proof. Now, consider the special case

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&= -\sum_{i \neq 1} \frac{\partial f}{\partial x^i} dx^i \wedge dx^J + \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^J
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\mathsf{V}\lrcorner\, d\varphi + d(\mathsf{V}\lrcorner\,\varphi) &= \frac{\partial}{\partial x^1}\lrcorner\, d\left(f dx^1 \wedge dx^J\right) + d\left(f dx^J\right) \\
&= -\frac{\partial}{\partial x^1}\lrcorner\, \left(dx^1 \wedge df \wedge dx^J\right) + df \wedge dx^J \\
&= -\sum_{i\neq 1} \frac{\partial f}{\partial x^i} dx^i \wedge dx^J + \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^J \\
&= \frac{\partial f}{\partial x^1} dx^1 \wedge dx^J
\end{aligned}$$

Cartan's Magic Formula: If $\varphi \in \Omega^k(M)$, then, for any $\mathsf{V} \in \mathfrak{X}(M)$, then

$$\boxed{\mathcal{L}_{\mathsf{V}}\varphi = \mathsf{V} \lrcorner\, d\varphi + d(\mathsf{V} \lrcorner\, \varphi).}$$

Proof. Next, consider the special case

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Exercise: RHS is actually bilinear over $C^\infty(M)\ldots$

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Induction!

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Definition.

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Definition. *A smooth n-manifold*

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is trivial as a vector bundle.